Optimal Allocation, Input-output Linkages, and Aggregate Productivity Growth

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Abstract

We characterize the optimal allocation of resources across sectors and derive a sufficient statistics for allocative efficiency, which allows us to decompose the observed aggregate productivity growth in the data into the improvement in allocative efficiency and that in the fundamental technology. Using the KLEMS data set and the World Input-output Table, we show that allocative efficiency can go a long way in explaining both the slowdown in productivity growth in the US and the widening Canada-US productivity gap. Although the inclusion of input-output linkages in the model amplifies the measured loss from misallocation, it does not significantly affect the trend in the allocative efficiency over time. We prove that this is partly due to the assumption needed to identify the parameters in the production functions.

Keywords: Multi-sector models; Optimal allocation across sectors; Input-output linkages; Productivity; Slowdown of productivity growth; Canada-US productivity gap

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1 Introduction

The trends in aggregate productivity around the world have attracted attention from both academic and policy circles. Economists often discuss the slowdown in aggregate productivity growth in major countries in the context of the "secular stagnation" debate (Summers, 2018). In the United States, aggregate productivity growth measured in both labor productivity and TFP, has slowed down before the 2007-09 financial crisis (Panel A of Figure 1 and Cette, Fernald and Mojon, 2016).[1] Although one potential explanation is the increasing difficulty of measuring output and capital in the era of rapid digitization, papers including Byrne, Fernald and Reinsdorf (2016) and Syverson (2017) show that, given the magnitude of the slowdown, measurement errors can not be the only reason. Compared to the US, the Canadian economy fared even worse. As shown in Panel B of Figure 1, labor productivity growth in Canada has fallen behind that in the US over the past three decades. In 1985, Canadian workers earned 84 percent of the income of their American counterpart; in 2010, that number fell below 75 percent.[2]

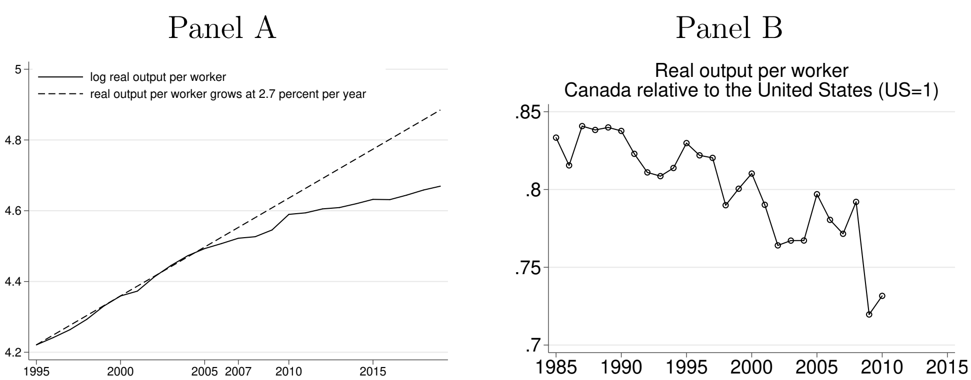

Figure 1: **Aggregate productivity in the US and Canada**

Source: BLS, FRED and PWT 9.1.
Notes: Panel A plots the logarithms of real output per work in the United States Business sector. Panel B B plots the real labor productivity of Canada relative to the US, where labor productivity is measured as real GDP per worker.

What are the reasons behind the slowdown of aggregate productivity growth and the widening gap between Canada and the US? We find that the trends in allocative

[1]The US economy also experienced a slowdown in growth during the 1970s (Vandenbroucke, 2019).

[2]We find a similar trend with real output per hour.

efficiency can go a long way to explain the puzzling trends in the US and Canada. As pointed out by Baqaee and Farhi (2019), the improvement in aggregate productivity can either comes from 1) a better production technology, that is, the expansion of productivity possibility frontier or 2) a better allocation of resources across production units, that is, the distance to the frontier. Naturally, it is crucial to investigate which channel contributed to these two patterns since they impose very different challenges to our economy.

In this paper, we aim to perform a decomposition exercise and study how the trends in allocative efficiency affect aggregate productivity in the US and Canada. In doing so, we revisit the misallocation literature and apply their concept of allocative efficiency to the decomposition. We characterize the optimal allocation problem of a benevolent social planner and derive sufficient statistics for allocative efficiency. In this regard, our framework is most similar to Oberfield (2013) and Monge-Naranjo, Sanchez and Santaeulalia-Llopis (2019), where the authors characterize the planner's optimal allocation problem across firms and countries, respectively. Different from them, our framework focuses on the cross-sector and within-country allocation and take into account the input-output linkages. Our approach is different from, but closely related to the decentralized approach used in Hsieh and Klenow (2009) and Jones (2013), which explicitly models the "wedges" faced by the producers and uncover their distribution from the data.

The other and perhaps the more important goal of our paper is to clarify the constraints and limitations regarding the measurement of allocative efficiency and show the importance of treating measurement as an integral part of the theory. Measurement of allocative efficiency is an extremely challenging task, not only because of measurement errors, as pointed out by Bils, Klenow and Ruane (2020), but also because the identification of the parameters in the model is only feasible under some explicit or implicit assumptions.[3] Although it seems that the assumptions are a separate issue from the theory, they govern the sources of misallocation that we could or could not uncover in the data. In our baseline result, the sufficient statistics of allocative efficiency is a function of the sectors' share of capital, labor and intermediate goods in the data and the output elasticities in the production functions, which we identify under the as-

[3] For example, in Hsieh and Klenow (2009), the author assumes that there is no distortion in the US economy. Therefore the US data can be used to identify the parameters in the production functions.

sumption that the distortions in the data are not input-specific.[4] We prove that under this identification assumption, the allocation of the *expenditure* of capital, labor, and intermediate goods in the data is always optimal, but the allocation of the *quantities* of these inputs are not necessarily so. In other words, the misallocation uncovered in our baseline result can only come from the dispersion in implied *prices* of the inputs across sectors.

The above result has important implications regarding what we can uncover as “misallocation" in the data. First, because intermediate inputs are measured by their nominal value (expenditure) at the sectoral level, their allocation does not affect the measured allocative efficiency. Although our empirical result shows that the input-output linkages amplify the loss from misallocation, it should be clear that the amplification does not come from the allocation of intermediate goods. Second, many papers in the literature use labor compensation or wage bills as a proxy for quality-adjusted labor inputs (see Hsieh and Klenow, 2009). Under our identification assumption, however, the allocation of labor compensation is always optimal. Therefore, although the measured allocative efficiency is higher with labor compensation as the proxy for labor inputs, it is only a result of our identification assumption.

Our benchmark framework is an input-output economy with Cobb-Douglas production and consumption, two factors (capital and labor), and perfect competition. As a comparison, we also study an economy without the input-output linkages. As we mentioned previously, instead of a decentralized economy with distortionary wedges as in Jones (2013), we consider the planner's optimal allocation problem to characterize the optimal allocation and derive the sufficient statistics of allocative efficiency. Since the model admits an aggregate production function, it allows us to decompose aggregate labor productivity and TFP in the data into three distinctive components: 1) allocative efficiency, 2) aggregate productivity growth under optimal allocation and 3) a “mismeasurement" term due to distorted aggregate capital share in the data.[5]

Our framework misses two important features. First, the Cobb-Douglas production system dictates that the elasticity of substitution between inputs is one (Jones, 2013). We show, using a CES production system, that the measured allocative efficiency generally decreases with the elasticity of substitution (see also Epifani and Gancia, 2011 and Osotimehin and Popov, 2018). Our baseline result underestimates the role

[4]In other words, they are in the form of sector-level taxes and subsidies as in Jones (2013).

[5]The third term only appears in the decomposition of aggregate TFP.

played by allocative efficiency if the inputs are gross complements. Second, we abstract from markups and profits at the sectoral level. To address this issue, we establish an equivalence result that maps the allocation of an economy with imperfect competition and markups to that with perfect competition and distorted input prices. Once profits at the sectoral level are properly calculated, this equivalence result allows us to use our current framework to measure the allocation efficiency in the presence of heterogeneous markups across sectors.

For the empirical analysis, we use the 2013 version of KLEMS and World Input-output Table (WIOT) covering 28 private sectors. As such, our analysis is limited to the allocative efficiency across the sector. Ideally, one would use a data set that spans the universe of firms in a country to study both the allocation within and across sectors. However, firm-level data sets that span the entire economy are not available for the US, and we do not wish to sacrifice the macroeconomic implications by limiting to a subset of sectors. Second, input-output information is not available at the firm-level, only at the sector-level. Besides, compared to firm-level data, sector-level data arguably have few measurement issues that would result in biases in the measurement of allocative efficiency.

Our results show that, in the US, except for the 1970s and 2000s, allocative efficiency had improved gradually. During the 1970s and the 2000s, allocative efficiency was stagnant. The stagnation in allocative efficiency is an important reason behind the slowdown in productivity growth during these two decades. The allocative efficiency in Canada, on the other hand, has stayed relatively constant from 1985 to the middle of the 2000s and decreased afterward. Quantitatively, the lack of improvement in allocation in Canada relative to the United States can account for the widening productivity gap between the two countries entirely.

We find that the allocation of capital is the main driver behind the trends in allocative efficiency in both countries, whereas the allocation of labor stayed relatively unchanged. One explanation behind this phenomenon is that the development of the financial sector in the US improved capital allocation. Most notably, several service sectors in the US have gained capital over time, and they are much closer to the optimal level now than several decades ago. This pattern is consistent with the slow reallocation across sectors during the structural change from the manufacturing to the service sector.

Literature review

Our paper builds on the strand of literature that measures the aggregate loss from misallocation. Seminal contributions to this literature include Hsieh and Klenow (2009), and Restuccia and Rogerson (2008), Jones (2011), and Jones (2013). Our benchmark model is closest to Jones (2013), but instead of a decentralized problem, we solve the planner's optimal allocation problem and derive a sufficient statistics of allocative efficiency. In this regard, our approach follows Oberfield (2013), which uses firm-level data and does not include input-output linkages. Osotimehin and Popov (2018) extends Jones (2011) framework and show whether input-output linkages amplifies misallocation depends on the elasticity of substitution between intermediate inputs. Caliendo, Parro and Tsyvinski (2018) builds a model of the world economy with both domestic and international input-output linkages and derive sufficient statistics for the loss from both internal and external distortions.

Our paper is closely related to Basu and Fernald (2002) and Baqaee and Farhi (2019), where they decompose aggregate productivity growth into improvement in technology and allocative efficiency. Baqaee and Farhi (2019) uses first-order approximation, which allows flexibility to apply the formula without knowing the production functions and the form of the distortions. Our approach requires the specification of the production system, but the result does not rely on linearization and therefore is more accurate, mainly when applied to long-run productivity growth. The notion of allocative efficiency in our paper measures the distance from the production possibility frontier, and it is the same as the misallocation literature. As pointed out by Baqaee and Farhi (2019), their measure is the improvement in allocation given factor supplies and technologies. The two notions differ when there are changes in technology and factor supplies. In related literature, Oberfield (2013) and Osotimehin (2019) ask how much of the aggregate productivity fluctuation over the business cycle (or in a recession) can be accounted for by allocative efficiency.

The structure of the paper is as follows: in section 2, we characterize the optimal allocation and the decomposition framework. Section 3 presents the data, the identification assumption and the study the implications of the assumptions. Section 4 and section 5 present the results and the robustness check. Section 6 concludes.

2 Model

In this section, we analyze optimal allocation across sectors as a planner's problem without input-output linkages (section 2.1) and with input-output linkages (section 2.2).

2.1 Optimal allocation without input-output linkages

There are N sectors in the economy ($i = \{1, ..., N\}$). In year t, each sector produces an intermediate good $Y_{i,t}$ using capital, labor and a Cobb-Douglas production function

$$Y_{i,t} = A_{i,t} K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}}.$$

There is one final good Y_t, which is produced by aggregating all intermediate goods, such that

$$Y_t = \prod_{i=1}^{N} Y_{i,t}^{\theta_{i,t}},$$

in which $\sum_i \theta_{i,t} = 1$.

2.1.1 Planner's problem

The planner's problem is to allocate aggregate capital K_t and labor L_t into the N sectors to maximize the output of final good Y_t, such that,

$$\max\ Y_t = \prod_{i=1}^{N} Y_{i,t}^{\theta_{i,t}}, \text{s.t. } Y_{i,t} = A_{i,t} K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}},\ \sum_i K_{i,t} = K_t,\ \sum_i L_{i,t} = L_t$$

The following proposition characterizes the optimal allocation across sectors and the optimal output.

Proposition 1. *The optimal allocation of capital and labor in this economy is such that* $K_{i,t} = \chi_{i,t}^{k*} K_t$ *and* $L_{i,t} = \chi_{i,t}^{l*} L_t$*, where* $\chi_{i,t}^{k*} = \frac{\theta_{i,t}\alpha_{i,t}}{\sum_i \theta_{i,t}\alpha_{i,t}}$ *and* $\chi_{i,t}^{l*} = \frac{\theta_{i,t}(1-\alpha_{i.t})}{\sum_i \theta_{i,t}(1-\alpha_{i,t})}$.

Proof. See Appendix B.1. □

Intuitively, the optimal allocation requires equalization of the marginal product of capital and labor across sectors. The optimal share of capital and labor ($\chi^{k*}_{i,t}$ and $\chi^{l*}_{i,t}$) reflect the relative importance of capital and labor in the production of the final good.

Allocative efficiency We define allocative efficiency $\mathbf{E}_t$ as the ratio between output in the data and output under the optimal allocation,

$$\mathbf{E}_t = \frac{Y_t}{Y^*_t},$$

where Y^*_t is the optimal output and Y_t is the output in the data. It can be shown using Proposition 1 that

$$\mathbf{E}_t = \prod_{i=1}^{N} [(\frac{\chi^k_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}} (\frac{\chi^l_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}}]^{\theta_{i,t}}, \tag{1}$$

where $\chi^k_{i,t} = \frac{K_{i,t}}{K_t}$ and $\chi^l_{i,t} = \frac{L_{i,t}}{L_t}$ are the sector i's capital and labor as a share of aggregate K_t and L_t in the data, respectively.

2.2 Optimal allocation with input-output linkages

Similar to the previous section, each sector produces an intermediate good $Q_{i,t}$ using capital, labor, domestic and imported intermediate goods, such that

$$Q_{i,t} = A_{i,t}(K^{\alpha_{i,t}}_{i,t} L^{1-\alpha_{i,t}}_{i,t})^{1-\sigma_{i,t}-\lambda_{i,t}}(\prod_{j=1}^{N} d^{\sigma_{ij,t}}_{ij,t})(\prod_{j=1}^{N} m^{\lambda_{ij,t}}_{ij,t}),$$

where $d_{ij,t}$ is the domestic intermediate good j used by sector i, $m_{ij,t}$ is the imported intermediate good j used by sector i, $\sigma_{i,t} = \sum_{j=1}^{N} \sigma_{ij,t}$, and $\lambda_{i,t} = \sum_{j=1}^{N} \lambda_{ij,t}$.

There is a final good in the economy, produced by aggregating over the intermediate goods,

$$Y_t = \prod_{i} Y^{\theta_{i,t}}_{i,t},$$

where $\sum_{i=1}^{N} \theta_{i,t} = 1$.

The resource constraint on the intermediate good i therefore can be written as

$$Q_{i,t} = Y_{i,t} + \sum_{j=1}^{N} d_{ji,t},$$

and the total expenditure on imported goods is

$$X_t = \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{P}_{j,t} m_{ij,t},$$

where $\bar{P}_{j,t}$ is the price of imported intermediate good j relative to the final good.

2.2.1 Planner's problem

The planner's problem is to allocate aggregate capital K_t, aggregate labor L_t, intermediate goods $Y_{i,t}$, $d_{ij,t}$ and $m_{ij,t}$ such that the aggregate output net of imports $(Y - X)$ are maximized,

$$\begin{aligned}
\max_{\{K_{i,t}, L_{i,t}, d_{ij,t}, m_{ij,t}\}_{i,j=1}^{N}} \quad & Y_t - X_t = \prod_i Y_{i,t}^{\theta_{i,t}} - \sum_{i=1}^{N} \sum_{j=1}^{N} \bar{P}_{j,t} m_{ij,t} \\
\text{s.t.} \quad & Q_{i,t} = A_{i,t} (K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}} (\prod_{j=1}^{N} d_{ij,t}^{\sigma_{ij,t}}) (\prod_{j=1}^{N} m_{ij,t}^{\lambda_{ij,t}}), \\
& Q_{i,t} = Y_{i,t} + \sum_{j=1}^{N} d_{ji,t}, \ \sum_i K_{i,t} = K_t, \ \sum_i L_{i,t} = L_t.
\end{aligned}$$

The optimal allocation is characterized by the following proposition,

Proposition 2. *The optimal allocation of capital, labor and intermediate goods in the economy is* $K_{i,t}^* = \chi_{i,t}^{k*} K_t$, $L_{i,t}^* = \chi_{i,t}^{l*} L_t$, $d_{ij,t}^* = \gamma_{ij,t}^* Q_{j,t}^*$, $Y_{j,t}^* = \chi_{j,t}^{y*} Q_{j,t}^*$, *and* $m_{ij,t}^* = (\frac{\theta_{i,t}\lambda_{ij,t}}{\chi_{i,t}^{y*}}) \frac{Y_t^*}{\bar{P}_{j,t}}$ *such that*

1. $\chi_{i,t}^{k*} = \frac{\theta_{i,t}\alpha_{i,t}(1-\sigma_{i,t}-\lambda_{i,t})}{1-\sum_j \gamma_{ji,t}^*} / \sum_s \frac{\theta_{s,t}\alpha_{s,t}(1-\sigma_{s,t}-\lambda_{s,t})}{1-\sum_j \gamma_{js,t}^*}$, $\forall i \in \{1, ..., N\}$.
2. $\chi_{i,t}^{l*} = \frac{\theta_{i,t}(1-\alpha_{i,t})(1-\sigma_{i,t}-\lambda_{i,t})}{1-\sum_{j,t} \gamma_{ji,t}^*} / \sum_s \frac{\theta_{s,t}(1-\alpha_{s,t})(1-\sigma_{s,t}-\lambda_{s,t})}{1-\sum_j \gamma_{js,t}^*}$, $\forall i \in \{1, ..., N\}$.
3. $\{\chi_{i,t}^{y*}\}_{i=1}^{N}$ *solve the system of equations*

$$\frac{1}{\chi_{i,t}} = 1 + \frac{1}{\theta_{i,t}} \sum_s (\frac{\theta_{s,t}}{\chi_{s,t}} \sigma_{si,t}), i \in \{1, ..., N\}, \tag{2}$$

and

$$\gamma_{ij,t}^* = \frac{\theta_{i,t}\chi_{j,t}^{y*}}{\theta_{j,t}\chi_{i,t}^{y*}} \sigma_{ij,t}. \tag{3}$$

4. $\{Q^*_{i,t}\}_{i=1}^N$ *solve for the system of equations*

$$Q_{i,t} = \chi_{Qi,t}(\prod_{s=1}^{N} Q_{s,t}^{\sigma_{is,t}+\lambda_{i,t}\theta_{s,t}}), i \in \{1, ..., N\},$$

where $\chi_{Qi,t} = A_{i,t}[(\chi^{k*}_{i,t}K_t)^{\alpha_{i,t}}(\chi^{l*}_{i,t}L_t)^{1-\alpha_{i,t}}]^{1-\sigma_{i,t}-\lambda_{i,t}}(\prod_{j=1}^N \gamma_{ij,t}^{*\sigma_{ij,t}})[\theta_{i,t}\prod_s(\frac{\chi^{y*}_{s,t}}{\chi^{y*}_{i,t}})^{\theta_{s,t}}]^{\lambda_{i,t}}$ $\prod_{j=1}^N(\frac{\lambda_{ij,t}}{P_{j,t}})^{\lambda_{ij,t}}$.

Proof. See Appendix B.2. □

Allocative efficiency We define the allocative efficiency as the ratio between the output net of imports in the data and that under the optimal allocation, such that

$$\mathbf{E}_t = \frac{Y_t - X_t}{Y^*_t - X^*_t}.$$

It can be shown using proposition 2 that $\mathbf{E}_t$ can be written as a product of allocative efficiency of capital, labor, domestic and imported intermediate goods, and intermediate goods used for final good production, such that

$$\mathbf{E}_t = E^{kl}_t \cdot E^d_t \cdot E^m_t \cdot E^y_t, \tag{4}$$

- $E^{kl}_t = \prod_{i=1}^N(((\frac{\chi^k_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}}(\frac{\chi^l_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}})^{\sum_n \theta_{n,t}C_{ni,t}}$ is the allocative efficiency of capital and labor.
- $E^d_t = \prod_{i=1}^N(\prod_{j=1}^N(\frac{\gamma_{ij,t}}{\gamma^*_{ij,t}})^{\sigma_{ij,t}})^{\sum_n \theta_{n,t}C_{ni,t}}$ is the allocative efficiency of domestic intermediate goods.
- $E^m_t = \frac{1-\sum_{n=1}^N \frac{\theta_{n,t}\lambda_{n,t}}{\chi^y_{n,t}}}{1-\sum_{n=1}^N \frac{\theta_{n,t}\lambda_{n,t}}{\chi^{y*}_{n,t}}}$ is the allocative efficiency of imported intermediate goods.
- $E^y_t = \prod_{n=1}^N(\frac{\chi^y_{n,t}}{\chi^{y*}_{n,t}})^{\theta_{n,t}}\prod_{i=1}^N(\frac{\prod_s(\frac{\chi^y_{s,t}}{\chi^y_{i,t}})^{\theta_{s,t}}}{\prod_s(\frac{\chi^{y*}_{s,t}}{\chi^{y*}_{i,t}})^{\theta_{s,t}}})^{\lambda_{i,t}\sum_n(\theta_{n,t}C_{ni,t})}$ is the allocative efficiency of intermediate goods used in the final goods production.

where C_t is an $N \times N$ matrix defined as $C_t = (I - \Omega_t)^{-1}$ and $\Omega_t(i,j) = \sigma_{ij,t} + \lambda_{i,t}\theta_{j,t}$.

2.3 Decentralized problem

So far, we focus on studying the planner's allocation problem. Although we do not use the decentralized problem directly in our analysis, it is important conceptually to match the model with the data. We consider a non-negative allocation (K_i, L_i, Q_i, Y_i, d_{ij}, m_{ij}) that satisfies the following feasibility constraints

$$\sum_i K_i \leqslant K, \sum_i L_i \leqslant L, Y_i + \sum_{j=1}^{N} d_{ji} \leqslant Q_i.$$

The next proposition shows that this allocation can be decentralized using a distorted price system. We focus on the allocations where the feasibility constraints hold with equality due to the convexity of the problem.

Proposition 3. *Any non-negative allocation* $\{K_i, L_i, Q_i, Y_i, d_{ij}, m_{ij}\}$ *that satisfies the feasibility constraints can be decentralized using a set of distorted prices* $\{(1-\tau_i^y)P_i, (1+\tau_i)^k R, (1+\tau_i)^l w, (1+\tau_{ij}^d)P_j, (1+\tau_{ij}^m)\bar{P}_j, (1+\tau_i^f)P_i\}$*, such that,*

1. *Given the distorted prices,* $(K_i, L_i, d_{ij}, m_{ij})$ *solve the problem of sector* i *producer,*

$$\begin{aligned} \max_{K_i, L_i, \{d_{ij}, m_{ij}\}} \quad & (1-\tau_i^y)P_i A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} d_{i1}^{\sigma_{i1}} \dots d_{iN}^{\sigma_{iN}} m_{i1}^{\lambda_{i1}} \dots m_{iN}^{\lambda_{iN}} \\ & -(1+\tau_i^k)RK_i - (1+\tau_i^l)wL_i \\ & -\sum_{j=1}^{N} (1+\tau_{ij}^d)P_j d_{ij} - \sum_{j=1}^{N} (1+\tau_{ij}^m)\bar{P}_j m_{ij} \end{aligned}$$

2. *Given prices* P_i, Y_i *solve the problem of the final good producer*

$$\max_{Y_i} \prod_i Y_i^{\theta_i} - \sum_i (1+\tau_i^f)P_i Y_i$$

3. *Prices clear the markets,* $\sum_i K_i = K, \sum_i L_i = L, Q_i = Y_i + \sum_{j=1}^{N} d_{ji}$

Proof. See Appendix B.3. □

In fact, the distorted decentralized problem admit a distorted aggregate production function in the Cobb-Douglas form $Y_t = A_t K_t^{\alpha_t} L_t^{1-\alpha_t}$ where Y_t, K_t, L_t are output, capital stock and labor inputs in the data and α_t is usually computed as the capital income share. Using the aggregate production function, we can back out the TFP from the data as $A = \frac{Y_t}{K_t^{\alpha_t} L_t^{1-\alpha_t}}$.

2.4 Decomposition of aggregate productivity in the data

This section uses the theoretical results in the previous two sections and shows how to decompose the aggregate productivity in the data. The results are summarized in Proposition 4 (labor productivity) and Proposition 5 (TFP).

Proposition 4. *Aggregate labor productivity measured in the data* LP_t *can be decomposed into 1) allocative efficiency* $\mathbf{E}_t$ *and 2) aggregate labor productivity under optimal allocation* LP_t^**, such that*

$$LP_t = LP_t^* \mathbf{E}_t. \tag{5}$$

Proof. The proof can be found in Appendix B.4. □

This proposition allows us to decompose aggregate labor productivity in the data into allocative efficiency and aggregate labor productivity under optimal allocation.

Proposition 5. *Aggregate TFP in the data* A_t *can be decomposed into three components: 1) allocative efficiency* $\mathbf{E}_t$*, 2) TFP under optimal allocation* A_t^**, and 3) a mismeaurement term* $(\frac{K_t}{L_t})^{\alpha_t^* - \alpha_t}$*, such that*

$$A_t = A_t^* \mathbf{E}_t (\frac{K_t}{L_t})^{\alpha_t^* - \alpha_t}. \tag{6}$$

For the economy without input-output linkages, $\alpha_t^* = \sum_i \alpha_i \theta_i$*; for the economy with input-output linkages,* $\alpha_t^* = \sum_n (\sum_i (\alpha_{i,t}(1 - \sigma_{i,t} - \lambda_{i,t}) C_{ni,t}) \theta_{n,t})$.

Proof. The proof can be found in Appendix B.4. □

Compared with Proposition 4, the decomposition of aggregate TFP has one additional component $(\frac{K_t}{L_t})^{{\alpha_t}^* - \alpha_t}$, where α_t is the capital income share we use to compute TFP in the data ($A_t = \frac{Y_t}{K_t^{\alpha_t} L_t^{1-\alpha_t}}$). If the economy deviates from optimal allocation, capital share α_t measured in the data is different from α_t^*, which leads bias in the estimation of the aggregate TFP. In fact, we prove that capital income share in an economy without distortions is exactly the same as α^*. In other words, the term which we labeled as "mismeasurement" is because of distorted aggregate capital share in the data.

Proposition 6. *If there is no distortion in the economy, capital income share in the data is equal to* α^**, the capital output elasticity in the aggregate production function under optimal allocation.*

Proof. See Appendix B.6 □

3 Empirical strategy

In this section, we discuss the empirical strategy used to answer our questions. First, we describe the datasets used in the empirical exercise (section 3.1). Second, we discuss the assumptions that allow us to estimate the model parameters using the data (section 3.2).

3.1 Data

We use the 2013 version of the KLEMS dataset and the 2013 version of the world input-output table (WIOT) for Canada and the US. The US KLEMS dataset covers the period of 1947-2010, while the input-output table covers 1995-2011. We restrict our analysis with input-output linkages to the overlapping periods of the two datasets: 1995-2010.[6] The analysis without input-output linkages can span the longer period of 1947-2010.

The Canadian KLEMS dataset covers the period of 1962-2008, while the input-output table covers 1995-2011. It reports three types of capital separately: residential structure, ICT assets, and non-ICT assets, which is one key difference from the US KLEMS. As discussed in detail in Section 4.2, we need to modify our model by replacing capital K_t with a Cobb-Douglas aggregation of the three types of capital.

For both countries, we restrict our analysis to $N = 28$ private sectors in the economy, as shown in Table 1. We use the 2013 version of the KLEMS, where sectors can be matched perfectly with those in the WOIT.

[6]The US KLEMS dataset has a 2017 version that covers the period 1970-2013. However, the 2017 version of KLEMS uses the ISIC revision 4.0 system, which does not allow a straightforward mapping to the ISIC revision 3.1 system used in the WIOT.

Table 1: List of sectors

Sectors
AtB Agriculture hunting forestry and fishing
C Mining and quarrying
D Manufacturing
15t16 Food products beverages and tobacco
17t19 Textiles textile products leather and footwear
20 Wood and products of wood and cork
21t22 Pulp paper paper products printing and publishing
23 Coke refined petroleum products and nuclear fuel
24 Chemicals and chemical products
25 Rubber and plastics products
26 Other non-metallic mineral products
27t28 Basic metals and fabricated metal products
29 Machinery nec
30t33 Electrical and optical equipment
34t35 Transport equipment
36t37 Manufacturing nec; recycling
E Electricity gas and water supply
F Construction
G Wholesale and retail trade
50 Wholesale trade and commission trade except of motor vehicles and motorcycles
51 Sale maintenance and repair of motor vehicles and motorcycles; retail sale of fuel
52 Retail trade except of motor vehicles and motorcycles; repair of household goods
H Hotels and restaurants
I Transport and storage and communication
60t63 Transport and storage
64 Post and telecommunications
J Financial intermediation
K Real estate, renting and business activities
70 Real estate activities
71t74 Renting of m&eq and other business activities
M Education
N Health and social work

Note: The sectors marked red are included in the analysis.

3.2 Identification of parameters

As shown in Proposition 1 and 2, the calculation of the allocative efficiency $\mathbf{E}_t$ requires 1) the output elasticities in the production functions $\alpha_{i,t}$, $\sigma_{ij,t}$, $\lambda_{ij,t}$ and $\theta_{i,t}$, and 2) each sector's capital and labor as a share of aggregate capital and labor $\delta^k_{i,t}$ and $\delta^l_{i,t}$. This section shows how we back out these parameters from the data.

Output elasticities The previous literature on misallocation often make the assumption that the US economy is under optimal allocation, therefore the output elasticities in the production functions are equal to the inputs expenditure shares in the data for the United States. However, since we want to study misallocation in the US economy, we are not able to make the same assumption and therefore need to make the following assumption for identification:

Assumption 7. (**Identification assumption**) The distortions faced by each sector are sector-specific and not inputs specific. In other words, the distortions in the data are in the form of sectoral tax or subsidy as in Jones (2011).

Under this assumption, capital, labor and intermediate expenditure income shares are not distorted in the data. More formally, the output elasticities in the production functions can be backed out by the expenditure shares in the data:

$$\begin{aligned}
\alpha_{i,t}(1-\sigma_{i,t}-\lambda_{i,t}) &= \frac{R_{i,t}K_{i,t}}{R_{i,t}K_{i,t}+w_{i,t}L_{i,t}+\sum_j p_{j,t}d_{ij,t}+\sum_j \bar{p}_{j,t}m_{ij,t}}, \\
(1-\alpha_{i,t})(1-\sigma_{i,t}-\lambda_{i,t}) &= \frac{w_{i,t}L_{i,t}}{R_{i,t}K_{i,t}+w_{i,t}L_{i,t}+\sum_j p_{j,t}d_{ij,t}+\sum_j \bar{p}_{j,t}m_{ij,t}}, \\
\sigma_{ij,t} &= \frac{p_{j,t}d_{ij,t}}{R_{i,t}K_{i,t}+w_{i,t}L_{i,t}+\sum_j p_{j,t}d_{ij,t}+\sum_j \bar{p}_{j,t}m_{ij,t}}, \\
\lambda_{ij,t} &= \frac{\bar{p}_{j,t}m_{ij,t}}{R_{i,t}K_{i}+w_{i,t}L_{i,t}+\sum_j p_{j,t}d_{ij,t}+\sum_j \bar{p}_{j,t}m_{ij,t}}, \\
\theta_{i,t} &= \frac{p_{i,t}Y_{i,t}}{\sum_i p_{i,t}Y_{i,t}}.
\end{aligned}$$

All the components of RHS of the above equations can be found in the data.

- $R_{i,t}K_{i,t}$ is the capital compensation of sector i in year t (KLEMS).
- $w_{i,t}L_{i,t}$ is the labor compensation of sector i in year t (KELMS).
- $p_{j,t}d_{ij,t}$ is sector i's spending on domestic intermediate good from sector j (WIOT).
- $\bar{p}_{j,t}m_{ij,t}$ is sector i's spending on imported intermediate good from sector j (WIOT).
- $p_{i,t}Y_{i,t}$ is sector i's good used in final good production, which is computed as the gross output of sector i net of good i as intermediate inputs, that is, $p_{i,t}Y_{i,t} = p_{i,t}Q_{i,t} - p_{i,t}\sum_j d_{ji,t}$ (KLEMS and WIOT).

Sector's capital and labor as a share of aggregate capital and labor We measure capital $K_{i,t}$ using real capital (in the case of Canadian data, by three types of capital) and labor $L_{i,t}$ using the number of workers. The aggregate capital and labor stock K_t and L_t in the economy are the sums of sectoral level capital and labor stock, and we can compute each sector's capital and labor as a share of aggregate capital and labor $\delta^{k}_{i,t}$ and $\delta^{l}_{i,t}$.

- $K_{i,t}$ is the real capital stock in sector i and year t.
- $L_{i,t}$ is hours or labor compensation in sector i and year t.

3.3 Measurement under Identification Assumption 7

We next discuss the implications of Assumption 7 on the measurement of allocative efficiency. It turns out that under this identification assumption, the allocation of expenditure of the inputs across sectors in the data are equal to the optimal allocation. But the allocation of their quantity across sectors are not necessarily optimal. More precisely, recall in Proposition 2, the optimal allocation of labor

$$\chi^{l*}_{i,t} = \frac{\theta_{i,t}(1-\alpha_{i,t})(1-\sigma_{i,t}-\lambda_{i,t})}{1-\sum_{j,t}\gamma^{*}_{ji,t}} / \sum_{s} \frac{\theta_{s,t}(1-\alpha_{s,t})(1-\sigma_{s,t}-\lambda_{s,t})}{1-\sum_{j}\gamma^{*}_{js,t}}.$$

We can show that under Assumption 7,

$$\begin{aligned} \chi^{l*}_{i,t} &= \frac{w_{i,t}L_{i,t}}{\sum_i w_{i,t}L_{i,t}} \\ &= \frac{L_{i,t}}{\sum_i L_{i,t}} \text{ if } w_i = w. \end{aligned}$$

There are two interpretations for the result. First, if we assume relative expenditure share of the inputs are not distorted and use them to identify output elasticity in the sectoral production functions, then the expenditure of the inputs can not be used to identify misallocation, only the quantity of these inputs can. Second, if we use quantity of the inputs to calculate allocation efficiency, we assume implicitly that the source of misallocation is the difference in the implied price of inputs across sectors.[7]

In our baseline result, the allocative efficiency $\mathbf{E}_t$ is only determined by the allocative efficiency of capital and labor, because the intermediate goods are measured using their expenditure instead of quantity in the data, in sum,

[7] We refer to them as “implied" prices because we do not observe them directly in the data. The implied prices of an input are different across sectors when the distribution of its expenditure across sectors are different from the distribution of its quantity.

Proposition 8. *Under Assumption 7 and if intermediate goods are measured using expenditure, allocative efficiency with input-output linkages reduces to*

$$\mathbf{E}_t = E_t^{kl} = \prod_{i=1}^{N}(((\frac{\chi_{Ki,t}}{\chi^*_{Ki,t}})^{\alpha_{i,t}}(\frac{\chi_{Li,t}}{\chi^*_{Li,t}})^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}})^{\sum_n \theta_{n,t} C_{ni,t}}. \tag{7}$$

Proof. See Appendix B.5. □

Two results follow from this above proposition. First, because the allocation efficiency of intermediate goods (both domestic and foreign) do not enter the above equation, they are not the reason behind the differences in measured allocative efficiency in the model with and without input-output linkages.

Second, Jones (2011) and Jones (2013) provides an insight that input-output linkages could amplify the loss from misallocation. Using our language, it means that the measured allocative efficiency could be lower with input-output linkages than without them. We show in the following proposition that his insight holds in theory.

Proposition 9. *Under Assumption 7, the following inequality holds,*

$$\begin{aligned}\mathbf{E}_t^{woio} &= \prod_{i=1}^{N}[(\frac{\chi^k_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}}(\frac{\chi^l_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}}]^{\theta_{i,t}} \\ &> \prod_{i=1}^{N}(((\frac{\chi_{Ki,t}}{\chi^*_{Ki,t}})^{\alpha_{i,t}}(\frac{\chi_{Li,t}}{\chi^*_{Li,t}})^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}})^{\sum_n \theta_{n,t} C_{ni,t}} = \mathbf{E}_t^{wio},\end{aligned}$$

where $\mathbf{E}_t^{woio}$ *denotes measured allocative efficiency without input-output linkages and* $\mathbf{E}_t^{wio}$ *denotes measured allocative efficiency with input-output linkages.*

Proof. See Appendix B.7. □

4 Results

In this section, we show empirical results on how trends allocative efficiency across sectors can explain the slowdown of aggregate productivity growth in the US (section 4.1) and the widening Canada-US productivity gap (section 4.2). In section 4.3, we explore which sectors and inputs can explain the trends in the aggregate allocative efficiency in these two countries.

4.1 Slowdown of productivity growth in the US

In this section, we study the evolution of allocative efficiency in the US and show how it contributed to aggregate productivity growth.

4.1.1 Allocative efficiency over time

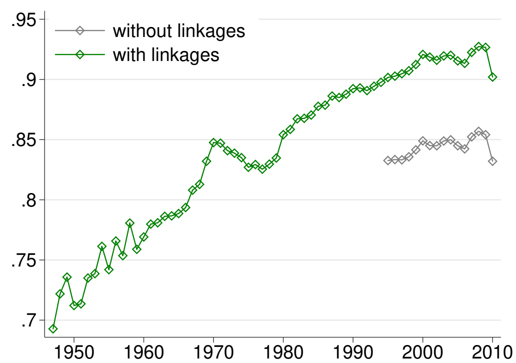

Figure 2: **Allocative efficiency in the US over time**

Note: In this figure, capital K is measured using real capital stock and L is measured using the number of workers. The black line corresponds to the model with the input-output linkage and the blue line is the one without the linkage.

Figure 2 display the allocative efficiency with and without input-output linkages over time. Since solving the allocation problem without input-output linkages does not require information about the input-output structure, the result goes back to 1947. Measured allocative efficiency is higher without input-output linkages, which suggests that missing the linkages leads to an underestimate of the loss from misallocation. Despite the difference in level, the two lines show very similar trends for the period 1995-2010. As discussed previously, this could be a result from our identification assumptions, under which the allocative efficiency of intermediate input goods are always 1 and thus does not affect the measured $\mathbf{E}_t$.

4.1.2 Slowdown of productivity growth in the 2000s

We first ask whether allocation can explain the slowdown in aggregate TFP growth in the US documented in Panel A of Figure 1.

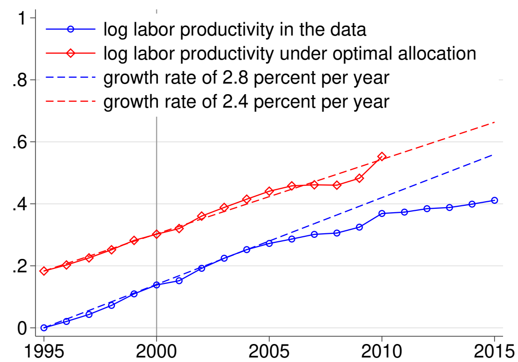

Figure 3: **Labor productivity in the data and under optimal allocation**

Source: BLS, FRED, KLEMS, WOIT, authors' own calculation.
Note: Labor productivity is calculated as real output per worker.

In Figure 3, we plot the aggregate productivity growth observed in the data A^{data} and that under optimal allocation A^*. There are two main takeaways in this figure. First, in the data, aggregate TFP grows at approximately 1.55 percent per year during the period 1995-2000 but only grows at 1 percent per year under the optimal allocation. The improvement in allocative efficiency accounts for about one-third of the aggregate productivity growth during this period. Second, there is a clear productivity growth slowdown in the data after 2005. However, this slowdown in productivity growth disappears under the optimal allocation. This result shows that the fundamental productivity growth has not slowed down at all. The lack of improvement in allocative efficiency can account for entirely the slow down in aggregate productivity growth after 2005.

As shown in Figure A.1 in the Appendix, aggregate TFP shows a similar trend as aggregate labor productivity in Panel A of Figure 1: both series display a break at around 2005 and significantly slowed down afterward. Unfortunately, we are unable to decompose the aggregate TFP due to the limitations of the data. For the BLS measure of aggregate TFP in A.1, we are missing the detailed information of the sectoral level capital and labor share. On the other hand, for the KLEMS data, which provides sector-level capital and labor share, there is no good measure of aggregate TFP for the 28 private sectors used in our analysis.

In our previous analysis concerning labor productivity, we take a static view and abstract from the impact of allocation on capital accumulation. This simplification is problematic when decomposing aggregate TFP. To see this, recall the decomposition equation in Proposition 4 and 5, we can rewrite the equation as the following

$$\underbrace{\log(LP_t^*) - \log(LP_t)}_{\text{labor productivity gap}} = -\log(\mathbf{E}_t),$$

$$\underbrace{\log(A^*) - \log(A)}_{\text{TFP gap}} = -\log(\mathbf{E}_t) - (\alpha_t^* - \alpha_t)\log(\frac{K_t}{L_t}).$$

The first equation shows that the gap between the "true" labor productivity and measured labor productivity, taken as given aggregate total capital K_t and labor L_t, is only a function of the allocative efficiency ($\mathbf{E}_t$). However, the gap between "true" TFP and measured TFP, as shown in the second equation, is also affected by capital labor ratio $\frac{K_t}{L_t}$ when $\alpha_t^* - \alpha_t \neq 0$. As such, the growth rate in the capital labor ratio could affect the percent change in the TFP gap. In addition, since $\frac{K_t}{L_t}$ is not a unit-less measure, any change in the unit of capital or labor would lead to the change in the level of the TFP gap.

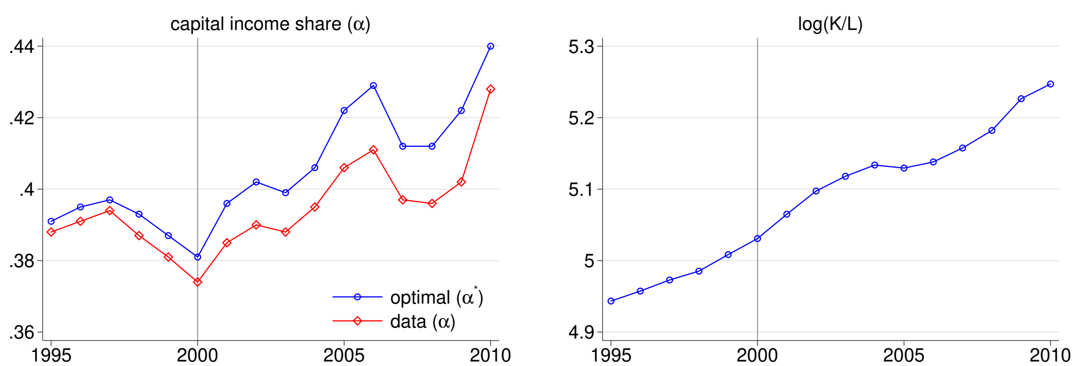

Figure 4: **Capital income share and capital labor ratio**

As shown in Panel A of Figure 4, capital share under optimal allocation (α_t^*, blue line) is higher than that in the data (α_t, red line) for all years and the gap between the two is widening after 2000. In other words, the $\alpha_t^* - \alpha_t$ is positive and increasing after 1995. It is well-known that capital income share has been increasing after 2000, or equivalently, the labor income share has been declining in the data, our result shows that the increase in capital income share is more prominent under the optimal allocation. On the other hand, the logarithms of capital-labor ratio, as shown in Panel B of Figure 4, also increased over time, but its growth slowed during the financial crisis. Therefore, it is clear that since $\alpha_t^* - \alpha_t \neq 0$, a change in capital-labor ratio could lead to a change in the measured TFP gap.

4.1.3 Slowdown of productivity growth in the 1970s

The labor productivity growth in the US slowed down significantly during the 1970s. This fact was documented and studied in many papers (see Vandenbroucke, 2019 for a summary of the literature). Figure 5 shows the productivity growth slowdown in both the logarithms of labor productivity (blue line) and the growth rate (red line).

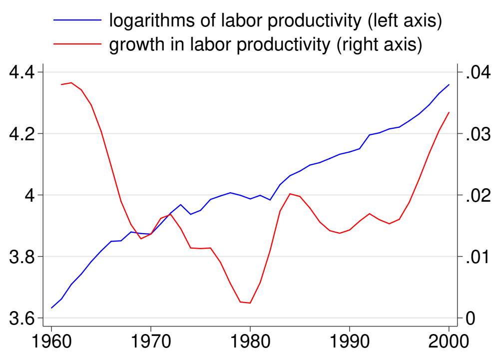

Figure 5: **Slowdown in labor productivity growth in the 1970s**

Source: BLS, FRED
Note: Labor productivity is measured as real output per worker. The growth rate is calculated as the log difference in labor productivity and HP-filtered with $\lambda = 6.25$.

Here, we provide a different explanation based on allocative efficiency. As discussed in section 4.1.1 and shown in Figure 2, there is a lack of improvement in allocative efficiency in the 1970s compared to the other decades, which could potentially explain the slowdown in productivity growth.

As shown in the second column of the Table 2, the growth in labor productivity is at 13 percent for the decade of 1970-1979, which is significantly lower than the 24 percent growth in the 1960s, 15 percent growth in the 1980s, and the 19 percent growth in the 1990s. The growth rate is 12 percentage points lower in the 1970s compared to the 1960s (third column).

The growth in labor productivity under optimal allocation, however, tells a slightly different story. Under optimal allocation, the increase in labor productivity is 13 percent in the 1970s, the same as the data because there is a minimal change in the allocative efficiency in the 1970s. However, for the other decades, the growth rates under optimal allocation are significantly lower than the ones in the data, which is consistent with the improvement in allocative efficiency. As a result, the slowdown in labor productivity of the 1970s compared with the 1960s is only eight percentage points. In other words, about one-third of the slowdown in the data could is due to movements in allocative efficiency.

Table 2: **Growth in labor productivity**

	Data		Optimal	
	dy/y	$\Delta dy/y$	dy/y	$\Delta dy/y$
1960-69	+0.24		+0.21	
1970-79	+0.13	-0.12	+0.13	-0.08
1980-89	+0.15	+0.02	+0.12	-0.01
1990-99	+0.19	+0.04	+0.15	+0.03

Source: BLS, FRED, Author's own calculation
Note: This table shows the growth rate and the changes in growth rate of labor productivity for different periods, both in the data and under optimal allocation. dy/y is the growth in labor productivity, measured in log differences and $\Delta dy/y$ is the change in the growth compared to the previous period. Labor productivity is computed as real output per worker.

4.2 The widening Canada-US productivity gap

It is a well-documented fact that Canadians are less productive than Americans. There exists a sizable gap between labor productivity in Canada and the US (see Sharpe, 2003 for a review). Several papers provided explanations for the productivity gap. Among them, Ranasinghe (2017) uses a quantitative model to show that differences in innovation costs account for the majority of firm size and productivity differences. Leung, Meh and Terajima (2008) uses an administrative firm-level data set to document that both labor productivity and TFP increase with firm size and that Canadian firms are less productive than their US counterparts because they are on average smaller.

In this section, instead of focusing on the level of the productivity gap, we look at the widening productivity gap between the two countries since the middle of the 1980s, as documented in Panel B of Figure 1.[8] In 1986, the Canadian labor productivity is around 85 percent that of the US, the number is now at approximately 75 percent.

[8] The gap narrowed during the 1970s, however, due the lack of sector-level data for the Canadian economy in the 1970s, we are unable to study this episode.

Figure 6 shows that the allocative efficiency in Canada is at approximately 90 percent in the middle of the 1980s, and it had remained relatively stable before 2000 and started to decline significantly since the beginning of the 2000s. On the contrary, the allocative efficiency in the US increased significantly before the 2000s and started to stabilize afterward.[9] There are two patterns that we observe in these two figures. First, allocative efficiency has improved faster in the US than in Canada. Second, there is a trend break at the beginning of the 2000s in both countries.

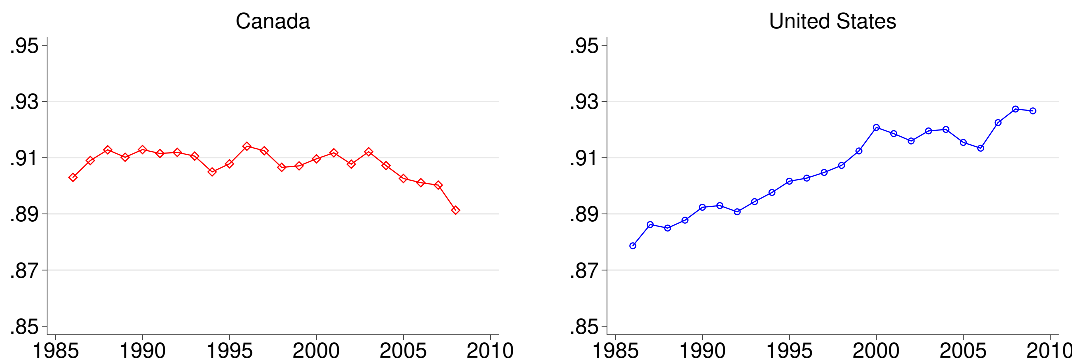

Figure 6: E_t **in Canada and US over time**

Source: KLEMS, author's own calculation.
Note: Allocation efficiency is calculated using KLEMS data without input-output linkages. K is measured using real capital stock and L as number of workers.

Figure 7 shows that under the optimal allocation, the productivity gap between Canada and the United States has been relatively constant since the middle of the 1980s. In other words, the widening productivity gap between Canada and the United States can be attributed entirely to the lack of improvement in allocation in Canada compared with the United States. If we interpret the aggregate productivity under optimal allocation as reflecting the fundamental technology of the economy, Figure 7 says that Canada has been able to keep up with the United States in terms of technology.

There are several caveats about the data that we should note here. First, the Canadian KLEMS reports different types of capital stock separately, whereas the US KLEMS aggregates different types of capital into one measure. In the analysis of the Canadian data, we need to replace K_t in the model by a Cobb-Douglas aggregation of

[9]Notice that the levels of the misallocation are not comparable across countries. The Canadian data separately report the different types of capital stock whereas it is not the case for the US data.

Figure 7: **Canada-US labor productivity gap under optimal allocation**

Source: KLEMS, BLS, FRED, author's own calculation.
Note: This figure plots the Canadian labor productivity under the optimal allocation relative to the US labor productivity under optimal allocation.

the four types. This difference is one reason why the level of allocative efficiency across countries is not comparable. We believe, however, the trend in allocative efficiency within each country is not subject to this caveat. Second, since the input-output data is only available after 1995, our analysis is restricted to the model without input-output linkages. However, as shown in Proposition 8 and Figure 2, the inclusion of input-output linkages in our model does not have a significant impact on the trend (growth rate) of measured allocative efficiency.

4.3 Capital, labor, and sectors

Next, we explore which sector and factor have contributed to the pattern observed in the data. To do this, we use the model without input-output linkages because it has a longer time series.

4.3.1 Capital and labor

The allocative efficiency can be decomposed into capital and labor allocative efficiency, such that

$$\mathbf{E}_t = \mathsf{E}^{k,t} \cdot \mathsf{E}^{l,t},$$

where $E^{k,t} = \prod_{i=1}^{N}[(\frac{\chi^{k}_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}\theta_{i,t}}$ and $E^{l,t} = \prod_{i=1}^{N}(\frac{\chi^{l}_{i,t}}{\chi^{l*}_{i,t}})^{(1-\alpha_{i,t})\theta_{i,t}}$ are the allocative efficiency of capital and labor, respectively. As shown in Figure 8, in the United States, capital allocative efficiency has improved drastically over the period of 1985 to 2000 whereas labor allocative efficiency has stayed stable. In contrast, there is no evidence of improvement in either capital or labor allocative efficiency in Canada. In fact, the allocative efficiency of capital has been declining after the beginning of 2000s.

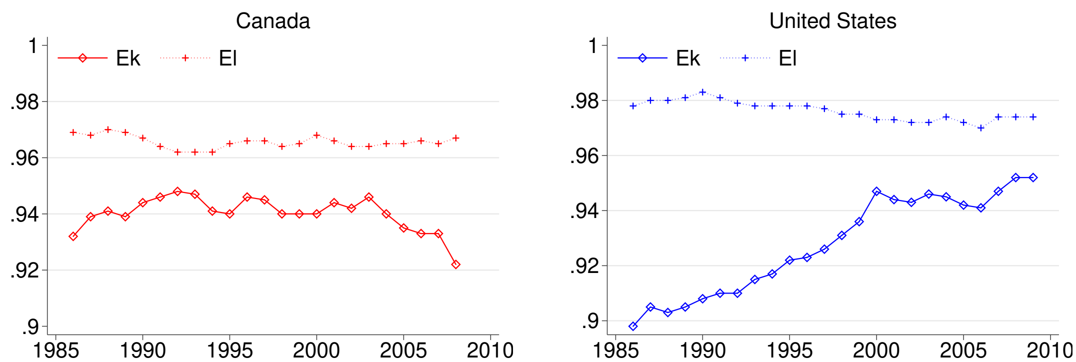

Figure 8: E^k **and** E^l **over time**

Source: KLEMS, authors' own calculation.
Note: This figure plots evolution of E_k and E_l in Canada and the US in a model without input-output linkages.

4.3.2 Sectors

Similarly, the aggregate allocative efficiency can be decomposed into sectoral allocative efficiency $E_{i,t}$, such that,

$$\mathbf{E}_t = \prod_{i=1}^{N} E_{i,t}^{\theta_{i,t}},$$

where $E_{i,t} = (\frac{\chi^{k}_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}}(\frac{\chi^{l}_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}}$. Figure 9 plots the distribution of the $E_{i,t}$ over time where different shades of colors represent different percentiles of the $E_{i,t}$ distribution in year t. Under the optimal allocation, sector-level allocative efficiency $E_{i,t} = 1$ for all sectors and the distribution collapses into one point at 1. If $E_{i,t} < 1$ for sector i, it means that capital and labor allocated to this sector is lower than the optimal level and vice versa.

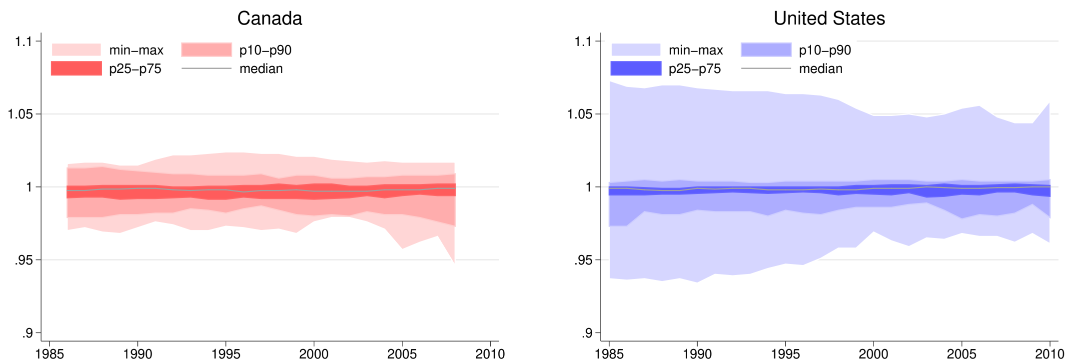

Figure 9: **Distribution of E_i over time**

Source: KLEMS, authors' own calculation.
Note: This figure plots the distribution of E_i in Canada and the US in a model without input-output linkages.

Several findings in this figure merit discussions. First, the distribution is more dispersed in the US than in Canada, which is consistent with the result in Figure 6 that the measured level of allocative efficiency is higher in Canada than the US. However, as discussed previously, the levels are not comparable across countries.

Second, there is a significant narrowing of the distribution in the US between the middle of 1985 and 2000, which is most visible among the industries that belong to the top and bottom ten percentile of the E_i distribution. In contrast, there are no significant changes in the distribution of Canada at the same time.

Third, after 2000, the distribution in the US stays relatively stable, whereas it becomes more dispersed in Canada. Notably, in Canada, the industries that belong to the bottom ten percentile in the distribution have moved further away from the optimal level.

Both Canada and the US experienced slowdown (or decline) in allocative efficiency after approximately 2000. Figure 10 plots the changes in E_i by sector in these two countries for the period before 2000 and that from 2000 to the beginning of the financial crisis. The circle and the cross represent the beginning and the end of each period, respectively. Therefore, the distance between the circle and the cross is the magnitude of the change. We mark the sectors green/black if their allocative efficiency has improved/deteriorated during this period (E_i moved closer to/further away from 1).

In the US, several service sectors experienced significant improvements in E_i during

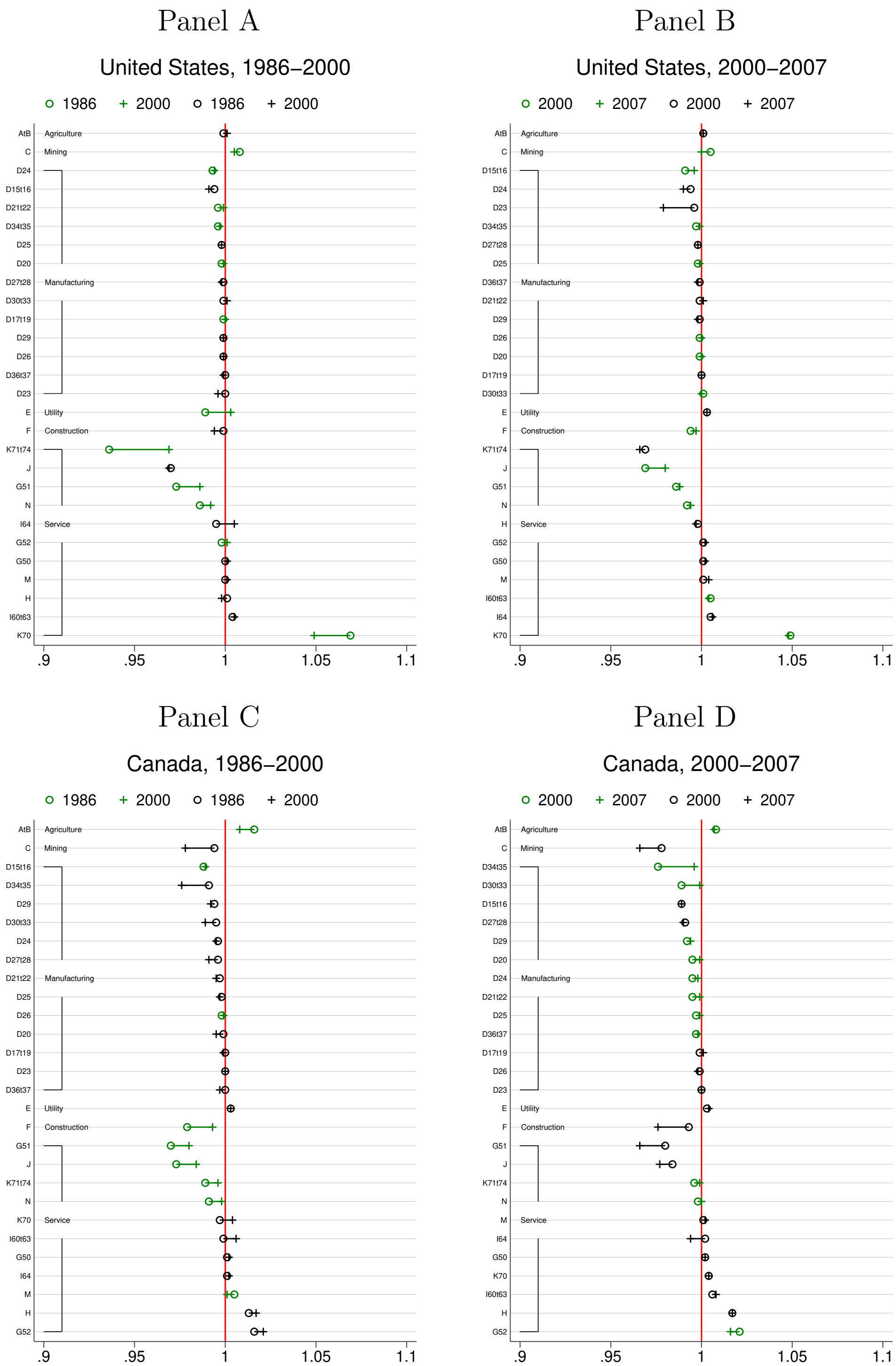

Figure 10: **Changes in E_i over 1986-2000 and 2000-2007**

Source: KLEMS, authors' own calculation.
Note: This figure plots evolution of E_k and E_l in Canada and the US in a model without input-output linkages.

1986-2000, most notably the sector of renting machinery and equipment and other business activities (K71t74), whereas there is very little change in the other sectors (Panel A). As shown in Panel B, during 2000-2007, the average improvement in E_i is much smaller in service sectors. Besides, the sector coke refined petroleum product and nuclear fuel (D23), experienced a significant decline in allocative efficiency. The E_i was close to 1 in 2000 but has decreased significantly in 2007, which indicates that the capital and labor allocated to this sector is lower than the optimal level.

In contrast, in Canada during 1986-2000, the service sectors also experienced some improvements in E_i. However, the magnitude is smaller than the US (Panel C). At the same time, the E_i of the mining sector (C) and the transport equipment sector (D34t35) deteriorated, which is perhaps the reason why aggregate allocative efficiency did not change significantly during this period. As shown in Panel D, during 2000-2007, a majority of the Canadian manufacturing sectors' E_i improved. However, several sectors, including mining (C), construction (F), sale, maintenance and repair of motor vehicles, retail sale of fuel (G51), and financial intermediation (J), experienced a significant decline in E_i. All five sectors' E_i were significantly below 1 in 2000, and they became even lower in 2007, which is consistent with the pattern in Figure 9 that the increase in the dispersion of the E_i distribution is driven by sectors in the bottom ten percentiles moving further away from the optimal level.

5 Robustness

In this section, we carry out several robustness checks. Sector 5.1 uses different measures of capital and labor to compute the allocative efficiency. Sector 5.2 extends our framework into a CES form. Section 5.3 discusses the inclusion of markups. One concern is the measurement errors in the KLEMS data, in sector 5.4, we restrict our analysis to the manufacturing sectors only. Lastly, our identification assumption that the distortion takes a specific form, in section 5.5, we explore the outcome with a relaxation of this assumption.

5.1 Alternative measure of capital and labor

So far, in our empirical analysis, we use real capital stock and the number of workers as the measure of capital and labor. One would argue that these measures do not contain information about the quality of labor and capital. Therefore the marginal product of capital and labor is mismeasured. Hsieh and Klenow (2009) uses wage bills as a proxy for labor inputs for firms, which they argue better captures the quality of labor. However, as we discussed in section 3.3, labor compensation and capital compensation can not be used to find misallocation under our identification Assumption 7.

We recall allocative efficiency in the model without input-output linkages as

$$\mathbf{E}_t = E^{k,t} \cdot E^{l,t} = \prod_{i=1}^{N} [(\frac{\chi_{i,t}^{k}}{\chi_{i,t}^{k*}})^{\alpha_{i,t}} (\frac{\chi_{i,t}^{l}}{\chi_{i,t}^{l*}})^{1-\alpha_{i,t}}]^{\theta_{i,t}}.$$

Figure 11 plots $E^{k,t}$ and $E^{l,t}$ with the different measures. There are two interesting patterns in this figure. First, in the baseline result (Panel A), the movement in E_t is driven mostly by that of capital, not labor, which is consistent with the results in Figure 8. Second, $E_k = 1$ when capital is measured using capital compensation and $E_l = 1$ when labor is measured using labor compensation. As shown in Panel D, aggregate allocative efficiency is always 1 if we use capital and labor compensation.

5.2 Elasticity of substitution

Our analysis so far is based on a Cobb-Douglas production system. In other words, the elasticity of substitution between the goods is assumed to be 1. One might wonder how the result changes with different degrees of elasticity of substitution. In this section, we modify the model without input-output linkages to a CES production system following Oberfield (2013).[10]

The final good is a CES aggregation of the N intermediate goods, such that

$$Y = (\sum_{i} \omega_i Y_i^{1-\frac{1}{\rho}})^{\frac{\rho}{\rho-1}}, \tag{8}$$

[10] We present the model without input-output linkages because of its simplicity. In appendix B.9, we extend the result to a CES production system with input-output linkages.

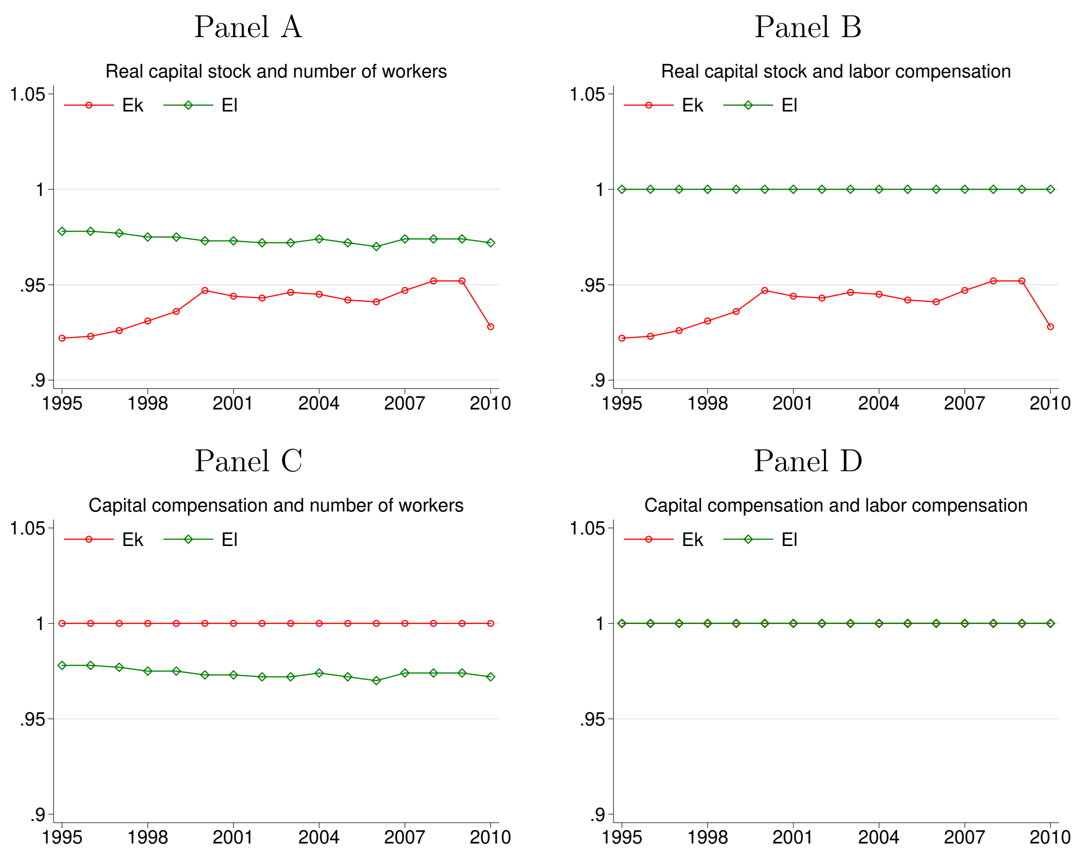

Figure 11: $\mathsf{E}_{k,t}$ **and** $\mathsf{E}_{l,t}$**, different measures of K and L**

Source: KLEMS, authors' own calculation.
Note: This figure plots $\mathsf{E}^{k,t}$ and $\mathsf{E}^{l,t}$ under different measures for capital and labor and without input-output linkages.

where ρ measures the elasticity of substitution and ω_i is the share of good Y_i in the final good production.[11]

The production function of the intermediate good Y_i is the Cobb-Douglas form, such that

$$Y_i = A_i K_i^{\alpha_i} L_i^{1-\alpha_i},$$

and the planner solves the following optimization problem

$$max\ Y, s.t \sum_i K_i = K, \sum_i L_i = L.$$

The following proposition characterizes the solution to the problem and the measured allocative efficiency.

Proposition 10. *The allocative efficiency* $\boldsymbol{E}_t$ *can be written as*

$$\boldsymbol{E}_t = \frac{Y_t^*}{Y_t} = \{\sum_j \{(\frac{P_j Y_j}{PY})^{\frac{\rho}{\rho-1}} (\frac{\alpha_j / K_j}{\bar{\alpha}/K})^{\alpha_j} [\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}, \tag{9}$$

where $\bar{\alpha}$ *is the solution to the following equation*

$$\bar{\alpha} = \sum_i \{\alpha_i \frac{\{(\frac{P_i Y_i}{PY})^{\frac{\rho}{\rho-1}} (\frac{\alpha_i}{\bar{\alpha}} \frac{K}{K_i})^{\alpha_i} (\frac{1-\alpha_i}{1-\bar{\alpha}} \frac{L}{L_i})^{1-\alpha_i}\}^{\rho-1}}{\sum_j \{(\frac{P_j Y_j}{PY})^{\frac{\rho}{\rho-1}} (\frac{\alpha_j}{\bar{\alpha}} \frac{K}{K_j})^{\alpha_j} [\frac{(1-\alpha_j)}{(1-\bar{\alpha})} \frac{L}{L_j}]^{1-\alpha_j}\}^{\rho-1}}\}. \tag{10}$$

In the equations, $P_i Y_i / PY$ *is expenditure share of good* i *in the final good production in the data,* K_i/K *and* L_i/L *are the shares of total capital and labor allocated to sector* i *in the data. The optimal allocation is characterized by* $\{\bar{\alpha}^*, \chi_i^{k*}, \chi_i^{l*}\}$*, such that*

$$\chi_i^{k*} = \frac{K_i^*}{K} = \frac{\frac{P_i Y_i}{PY}\alpha_i}{\bar{\alpha}^*} \quad \chi_i^{l*} = \frac{L_i^*}{L} = \frac{\frac{P_i Y_i}{PY}(1-\alpha_i)}{1-\bar{\alpha}^*},$$

and $\bar{\alpha}^* = \sum_i \frac{P_i Y_i}{PY} \alpha_i$.

Proof. See Appendix B.8. □

We show the impact of ρ on measured allocative efficiency in Figure 12. Measured allocative efficiency is higher when ρ is lower, that is, when the goods are less sub-

[11] In a monopolistic competition model, $\frac{\rho}{\rho-1}$ is the markups, which is the ratio of price over marginal cost.

stitutable (Panel A). This result is consistent with the finding in Epifani and Gancia (2011) and Osotimehin and Popov (2018), which show that the loss from misallocation increases when intermediate goods are more substitutable. To see why this is the case, we can rewrite equation 9 as the following

$$\mathbf{E}_t^{1-\rho} = \sum_j [\frac{P_j Y_j}{PY}(\frac{\bar{\alpha}^*}{\bar{\alpha}})^{\alpha_j(\rho-1)}(\frac{1-\bar{\alpha}^*}{1-\bar{\alpha}})^{(1-\alpha_j)(\rho-1)}(\frac{\chi_i^{k*}}{\chi_i^k})^{\alpha_j}(\frac{\chi_i^{l*}}{\chi_i^l})^{1-\alpha_j}]^{\rho-1}.$$

If we assume symmetry in the production system, that is, $\alpha_i = \alpha$ and $\frac{P_i Y_i}{PY} = \theta$, it can be shown using equation 10 that $\bar{\alpha} = \alpha$ for any data and $\bar{\alpha}^* = \alpha$. Therefore the above equation reduces to

$$\mathbf{E}_t^{1-\rho} = \sum_j [\theta(\frac{\chi_i^{k*}}{\chi_i^k})^{\alpha}(\frac{\chi_i^{l*}}{\chi_i^l})^{1-\alpha}]^{\rho-1},$$

where $\frac{\chi_i^{k*}}{\chi_i^k}$ and $\frac{\chi_i^{l*}}{\chi_i^l}$ measured the deviation of data allocation and optimal allocation. Following the methods in Epifani and Gancia (2011) and Osotimehin and Popov (2018), we can show that E_t decreases with ρ under the assumption that $\frac{\chi_i^{k*}}{\chi_i^k}$ and $\frac{\chi_i^{l*}}{\chi_i^l}$ follow independent log normal distributions.

In sum:

Proposition 11. *If 1) the production system is symmetric, and 2) $\frac{\chi_i^{k*}}{\chi_i^k}$ and $\frac{\chi_i^{l*}}{\chi_i^l}$ follow independent log normal distributions, measured allocative efficiency $\mathbf{E}_t$ decreases with the elasticity of substitution ρ.*

Panel A also shows that the growth of measured $\mathbf{E}_t$ is slower when ρ is lower. In panel B, a lower ρ is also associated with lower volatility in measured $\mathbf{E}_t$. There exists a range of estimates for ρ in the literature, most of which shows that $\rho > 1$, meaning goods are gross complements. If we take the estimate from Basu (1995) and set $\rho = 4.0$, it would suggest that our baseline result significantly underestimates the role played by allocation efficiency in explaining the aggregate productivity growth and during the 2007-09 financial crisis.

5.3 Markups

In our empirical analysis so far, we assume perfect competition. That is, sectors make zero profit. We make this assumption because the KLEMS data set does not distinguish

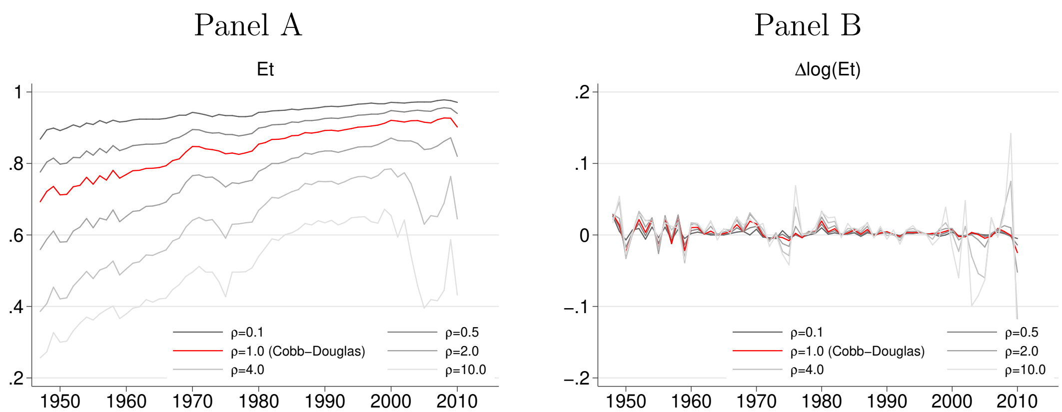

Figure 12: $\mathbf{E}_t$ **and** $\Delta\log(\mathbf{E}_t)$**, different** ρ

Source: KLEMS, authors' own calculation.
Note: This figure plots $\mathbf{E}_t$ and $\Delta\log(\mathbf{E}_t)$ under different value for ρ and without input-output linkages. Capital is measured using real capitals stock and labor is measured using the number of employers.

between profit and capital income. However, sector-level profits are not zero in reality, and recent papers suggest there is an increase in profit in the aggregate and at the sector-level in the US (Barkai, 2019 and Ahmad, Fernald and Khan, 2019).[12]

Including markups could change our results in two ways. First, estimated output elasticities in the production functions would change once we take the profit out of capital income in the KLEMS data. Second, dispersion in markups across sectors is an additional source of misallocation, which is currently missing in our baseline framework. In the remainder of this section, we address them in turn.

One common approach is to calculate the "required return to capital" following Hall and Jorgenson (1969). The required return to capital of sector i is $R_i P_i K_i$, where R_i is the cost of using capital, $P_i K_i$ is the nominal value of capital. For the purpose of our analysis, it is important to construct a sector-specific return series R_i rather than imposing that they are the same across sectors.[13]

We next show that our framework can be revised to study the economy with imperfect competition and markups. Consider the CES production system without input-output linkages in the previous session. Let the markup for sector i be $\mu_i = \frac{p_i}{MC_i}$ in

[12] Loecker and Eeckhout (2017) and Traina (2018) document an increase in firms' profit using firm-level data.

[13] For this version we do not perform such an exercise.

the data. In equilibrium the following equation holds

$$\frac{\omega_i Y_i^{1/\rho}}{\omega_j Y_j^{1/\rho}} = \frac{P_i}{P_j} = \frac{\mu_i MC_i}{\mu_j MC_j},$$

which shows that markups only distort the allocation of resources if they are different across sectors (Epifani and Gancia, 2011).

The allocation of the previous economy with imperfect competition and markups is equivalent to an environment of perfect competition and price distortions of the final good producer. In this case, the price of good i is MC_i, with which the intermediate good firm makes zero profit. The final good producer solves this following distorted problem,

$$\max_{Y_i} (\sum_i \omega_i Y_i^{1-\frac{1}{\rho}})^{\frac{\rho}{\rho-1}} - \mu_i MC_i Y_i,$$

where MC_i is the price of good i and μ_i is the wedge faced by final good producer when purchasing good i. The FOC of this problem reads

$$\frac{\omega_i Y_i^{1/\rho}}{\omega_j Y_j^{1/\rho}} = \frac{\mu_i MC_i}{\mu_j MC_j},$$

which is the same as the allocation in the economy with imperfect competition and markups. Notice, however, these two approaches differ regarding the distribution of profits in the economy. In the setting of markups, profit goes to the intermediate good producer, whereas we do not specify who gets the profit in the setting of wedges. Although this difference does not matter in our static model and with exogenous labor supply, it might affect the investment decisions and labor supply if we relax the assumptions of the model.

5.4 Manufacturing sector

One of the concerns regarding is the quality of the data and the existence of measurement errors. For example, it is challenging to measure intangible capital, and that might lead to biases in the measurement of allocation efficiency. In this section, similar to Hsieh and Klenow (2009) and Oberfield (2013), we focus on allocation across the 13

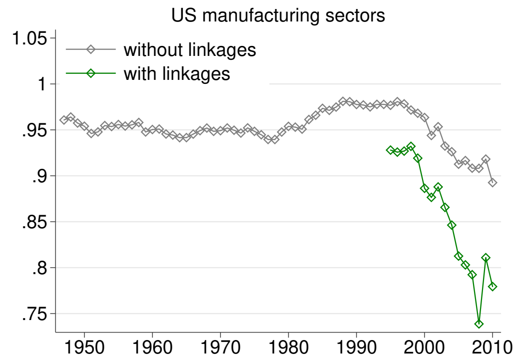

Figure 13: E_i **in the US manufacturing sectors over time**

Source: KLEMS, WIOT, authors' own calculation.
Note: This figure plots the allocative efficiency across the 13 manufacturing sectors in the US. Capital K is measured using real capital stock and L is measured using the number of workers. The black line corresponds to the model without input-output linkages and the green line is the one with input-output linkages.

manufacturing sectors (see Table 1), which arguably suffer less from the measurement error of intangible capital.

In Figure 13, we plot the allocative efficiency of the manufacturing sectors in the US over time. As shown in the black line (the model without input-output linkages), E_i of the manufacturing sectors was very stable before 1980; it increased slightly during the 1980s, stabilized again in the 1990s, and declined significantly after the beginning of the 2000s. The green line (model with input-output linkages) has a much short time series and also shows a significant decline during the 2000s, especially during the 2007-09 financial crisis.

In the next part of this section, we explore how the changes in allocative efficiency can explain the aggregate productivity in the manufacturing sector. In Panel A of Figure A.2, we plot the logarithms of labor productivity, measured as real output per worker in the manufacturing sector in the US. One of the most significant patterns of this graph is that the labor productivity growth rate is virtually zero after 2010. Unfortunately, our measure of allocative efficiency ends in 2010. Thus we do not have a way of telling whether allocative efficiency contributed to the aggregate labor productivity plateau in the manufacturing sector. In Panel B, we plot the lab productivity under optimal allocation during 1995 and 2010, which shows that during the 2000s, the growth of labor productivity in manufacturing under optimal allocation was significantly higher than the pre-2000 trend, a pattern different than that of total economy

(Figure 3).

5.5 Identification assumption

Our empirical analysis relies on the identification assumption 7, where we assume that the distortions are in the form of sectoral taxes and subsidies. The assumption implies that in our exercise, the output elasticities in the production functions change every year. In this section, we explore other identification assumptions and their impacts on the results.

In Hsieh and Klenow (2009), the authors assume that the US economy is under the optimal allocation, and thus, they can use the US data to identify the parameters in the production functions. As discussed previously, we can not make this assumption since our goal is to study the efficiency of allocation in the US. Instead, we pick a base year and assume that the allocation is optimal in that year.

In Figure 14, we plot the measured allocative efficiency under the assumption that there are no distortions in year 2005 (grey line) and under assumption 7 (blue line). The two lines intersected in 2005. However, compared with the baseline result, measured allocative efficiency is in general much lower under the assumption that the allocation is optimal in 2005.[14]

One interpretation of the above result is that the production functions of 2005 are not the same as those of the other years. Assuming that they are could lead to bias the measured allocative efficiency. Figure A.3 illustrates such a bias. Panel B of the figure shows that if 1960 is the base year, the measured allocative efficiency improved before the 1980s and has been declining ever since. This pattern is quite different from the baseline result, which suggests that the result is sensitive to the different identification assumptions, and we should interpret the results with caution.

[14] The baseline result is not necessarily the upper bound of allocative efficiency. We can prove that, for any allocation in the data, there exists a set of distortions and corresponding parameter values, under which the allocation is optimal.

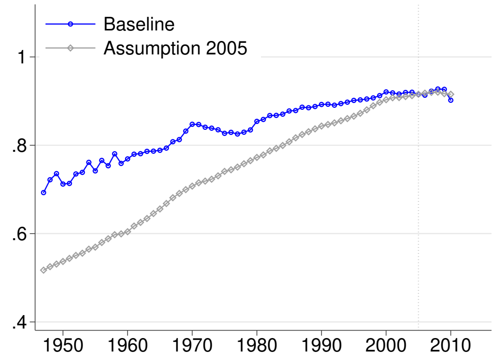

Figure 14: $\mathtt{E_i}$ **in the US, alternative identification assumptions**

Source: KLEMS, authors' own calculation.
Note: This figure plots the allocative efficiency (without input-output linkages) in the US under the assumption 7 (baseline) and the assumption that there is no distortions in 2005 (assumption 2005). Capital K is measured using real capital stock and L is measured using the number of workers.

6 Conclusion

In this paper, we show the importance of allocation in explaining the aggregate productivity growth. In theory, we derive a sufficient statistics for allocative efficiency and apply the theory to the US and Canada using the KLEMS and WIOT data sets. Our empirical analysis shows the importance of understanding the implications of the identification assumptions in the empirical analysis.

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Online Appendix

Not for Publication

A Additional tables and figures

A.1 Figures

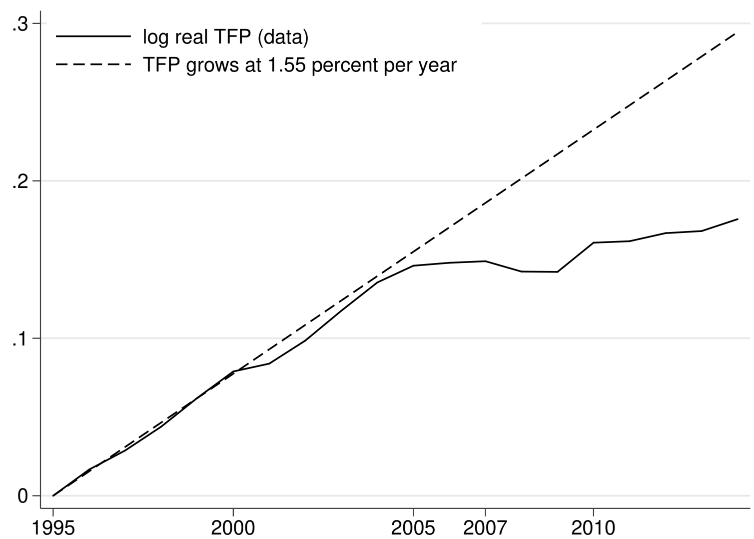

Figure A.1: **Aggregate TFP in the US**

Source: BLS, FRED
Notes: This figure plots the logarithms of real TFP of the United States Business sector.

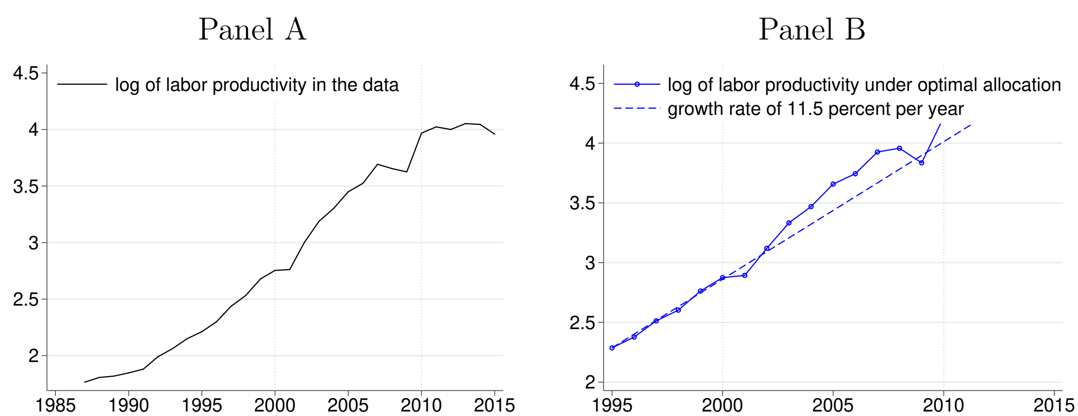

Figure A.2: **Labor productivity in the data and under optimal allocation, manufacturing sector**

Source: BLS, FRED, KLEMS, WIOT, authors' own calculation.
Note: This figure plots the measured allocative effiency (E_i) under different measures for capital and labor and with input-output linkages.

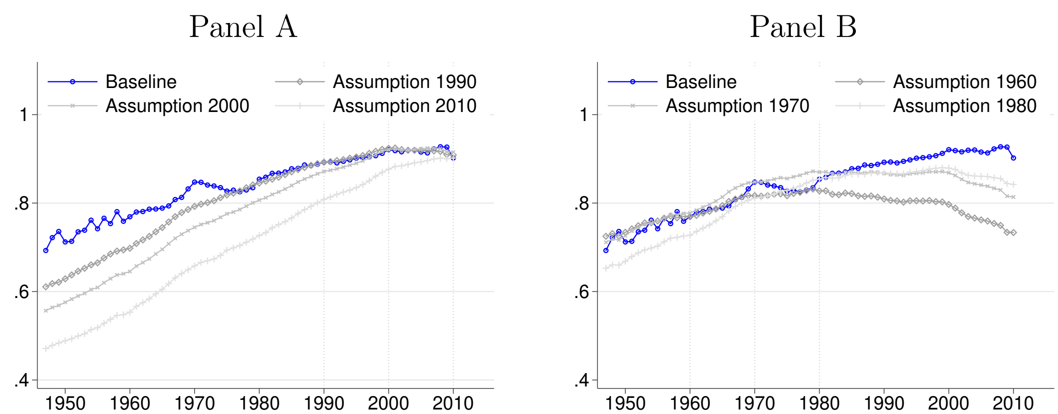

Figure A.3: E_i **in the US, alternative identification assumptions**

Source: KLEMS, authors' own calculation.
Note: This figure plots the allocative efficiency (without input-output linkages) in the US under the assumption 7 (baseline) and the assumption that there is no distortions in year 1960/1970/1980/1990/2000/2005/2010 (assumption 1960/1970/1980/1990/2000/2005/2010). Capital K is measured using real capital stock and L is measured using the number of workers.

B Proofs

In the proofs, we drop the time subscript t to simplify notation.

B.1 Proof of proposition 1

The optimal solution requires the equalization of MPK and MPL, such that,

$$\frac{\partial \log Y}{\partial K_i} = \lambda$$

$$\frac{\partial \log Y}{\partial L_i} = \eta$$

They can be written as,

$$K_i = \frac{\theta_i \alpha_i}{\lambda}$$

$$L_i = \frac{\theta_i (1-\alpha_i)}{\eta}$$

Given the resource constraint, we get

$$K_i = \frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i} K$$

$$L_i = \frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)} L$$

The final good output can be written as

$$\begin{aligned} Y &= \prod_i Y_i^{\theta_i} \\ &= \Pi_i (A_i K_i^{\alpha_i} L_i^{1-\alpha_i})^{\theta_i} \\ &= \Pi_i (A_i (\frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i} K)^{\alpha_i} (\frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)} L)^{1-\alpha_i})^{\theta_i} \\ &= \bar{A} K^{\sum_i \alpha_i \theta_i} L^{\sum_i (1-\alpha_i)\theta_i}, \end{aligned}$$

where $\bar{A} = \Pi_i (A_i (\frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i} K)^{\alpha_i} (\frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)} L)^{1-\alpha_i})^{\theta_i}$. *Q.E.D.*

B.2 Proof of proposition 2

The planner's problem is

$$C = \prod_{i=1}^{N}(Q_i - \sum_{j=1}^{N} d_{ji})^{\theta_i} - \sum_i \sum_j \bar{P}_j m_{ij}.$$

FOCs The first order conditions for K_i, L_i, d_{ij}, m_{ij} are

$$\begin{aligned}
\frac{\partial C}{\partial K_i} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{K_i}\alpha_i(1-\sigma_i-\lambda_i) \\
\frac{\partial C}{\partial L_i} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{L_i}(1-\alpha_i)(1-\sigma_i-\lambda_i) \\
\frac{\partial C}{\partial d_{ij}} &= \theta_i \frac{Y}{Y_i}[\frac{Q_i}{d_{ij}}\sigma_{ij} - I_{\{i=j\}}] + \theta_j \frac{Y}{Y_j}[\frac{Q_j}{d_{jj}}\sigma_{jj} I_{\{i=j\}} - 1] \\
\frac{\partial C}{\partial m_{ij}} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{m_{ij}}\lambda_{ij} - \bar{P}_j
\end{aligned}$$

The FOC $\frac{\partial C}{\partial d_{ij}} = 0$ implies

$$d_{ij} = \frac{\theta_i Y_j}{\theta_j Y_i}\sigma_{ij} Q_i, \tag{11}$$

therefore

$$\begin{aligned}
Y_j &= Q_j - \sum_{i=1}^{N} d_{ij} = Q_j - \sum_{i=1}^{N} \frac{\theta_i Y_j}{\theta_j Y_i}\sigma_{ij} Q_i, \\
Y_j[1 &+ \frac{1}{\theta_j}\sum_i (\frac{\theta_i Q_i}{Y_i}\sigma_{ij})] = Q_j.
\end{aligned}$$

Let $\chi_j = \frac{Y_i}{Q_i}$, $\{\chi_i\}_{i=1}^N$ solve the following equations

$$\frac{1}{\chi_i} = 1 + \frac{1}{\theta_i}\sum_s (\frac{\theta_s}{\chi_s}\sigma_{si}) \tag{12}$$

or

$$1 - \chi_j = \sum_i \sigma_{ij}\frac{\theta_i \chi_j}{\theta_j \chi_i}$$

Let $\gamma_{ij} = \frac{\theta_i \chi_j}{\theta_j \chi_i}\sigma_{ij}$ in equation 11, then $d_{ij} = \gamma_{ij} Q_j$. The market clear condition for Q_i implies

$$\chi_i = 1 - \sum_s \gamma_{si}.$$

The FOC $\frac{\partial C}{\partial m_{ij}} = 0$ implies

$$m_{ij} = \theta_i \frac{Y}{Y_i} Q_i \frac{\lambda_{ij}}{\bar{P}_j}$$

Since

$$Y = \prod_i Y_i^{\theta_i} = \prod_i (\chi_i Q_i)^{\theta_i}$$

we have

$$m_{ij} = \theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s} \prod_s (Q_s)^{\theta_s} \frac{\lambda_{ij}}{\bar{P}_j}$$

The FOC $\frac{\partial C}{\partial K_i} = 0$ and $\frac{\partial C}{\partial L_i} = 0$ lead to

$$K_i = \chi_{Ki} K$$

$$L_i = \chi_{Li} L$$

where

$$\chi_{Ki} = \frac{\frac{\theta_i \alpha_i (1-\sigma_i-\lambda_i)}{(1-\sum_j \gamma_{ji})}}{\sum_s \frac{\theta_s \alpha_s (1-\sigma_s-\lambda_s)}{(1-\sum_j \gamma_{js})}}, \chi_{Li} = \frac{\frac{\theta_i (1-\alpha_i)(1-\sigma_i-\lambda_i)}{(1-\sum_j \gamma_{ji})}}{\sum_s \frac{\theta_s (1-\alpha_s)(1-\sigma_s-\lambda_s)}{(1-\sum_j \gamma_{js})}}.$$

Solve for Q_i Rewrite production function of Q_i as

$$\begin{aligned} Q_i &= A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} (\gamma_{i1} Q_1)^{\sigma_{i1}} \cdots (\gamma_{iN} Q_N)^{\sigma_{iN}} \prod_{j=1}^{N} \{\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s} \prod_s (Q_s)^{\theta_s} \frac{\lambda_{ij}}{\bar{P}_j}\}^{\lambda_{ij}} \\ &= A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) (\prod_{j=1}^{N} Q_j^{\sigma_{ij}}) [\prod_s (Q_s)^{\theta_s}]^{\lambda_i} [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} \\ &= A_i [(\chi_{Ki} K)^{\alpha_i} (\chi_{Li} L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} (\prod_{s=1}^{N} Q_s^{\sigma_{is}+\lambda_i \theta_s}) \end{aligned}$$

Define

$$\chi_{Qi} = A_i [(\chi_{Ki} K)^{\alpha_i} (\chi_{Li} L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} \quad (13)$$

The above equation can be written as

$$Q_i = \chi_{Qi} (\prod_{s=1}^{N} Q_s^{\sigma_{is}+\lambda_i \theta_s}). \quad (14)$$

Taking log of equation 14 gives $\log Q_i = \log\chi_{Qi} + \sum_{j=1}^{N}[(\sigma_{ij} + \lambda_i\theta_j)\log(Q_j)]$. Let $x = [\log(Q_1), \ldots, \log(Q_N)]'_{N\times 1}$, equation 14 can be written as

$$x_{N\times 1} = b_{N\times 1} + \Omega_{N\times N} x_{N\times 1},$$

where $b(i) = \log\chi_{Qi}$ and $\Omega(i,j) = \sigma_{ij} + \lambda_i\theta_j$. Therefore x can be solved as $x = Cb$ where $C_{N\times N} = (I - \Omega)^{-1}$ and $Q_n = \prod_{i=1}^{N}(\chi_{Qi}^{C_{ni}})$.

Rewrite χ_{Qi} as

$$\chi_{Qi} = z_i K^{\alpha_i(1-\sigma_i-\lambda_i)} L^{(1-\alpha_i)(1-\sigma_i-\lambda_i)}$$

where $z_i = A_i({\chi_{Ki}}^{\alpha_i}{\chi_{Li}}^{1-\alpha_i})^{1-\sigma_i-\lambda_i}(\prod_{j=1}^{N}\gamma_{ij}^{\sigma_{ij}})[\theta_i\prod_s(\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i}\prod_{j=1}^{N}(\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}}$.

Then Q_n can be rewritten as

$$Q_n = \tilde{A}_n K^{\tilde{\alpha}_n} L^{\tilde{\beta}_n} \tag{15}$$

where $\tilde{A}_n = \{\prod_{i=1}^{N} {z_i}^{C_{ni}}\}$, and $\tilde{\alpha}_n = \sum_i(\alpha_i(1-\sigma_i-\lambda_i)C_{ni})$, $\tilde{\beta}_n = \sum_i((1-\alpha_i)(1-\sigma_i-\lambda_i)C_{ni})$.

Next we show that $\tilde{\alpha}_n + \tilde{\beta}_n = 1$. Let $B = I - \Omega$, following the definition of Ω

$$\sum_j B(i,j) = 1 - (\sigma_i + \lambda_i).$$

Since

$$\tilde{\alpha}_n + \tilde{\beta}_n = \sum_i (C_{ni}(1-\sigma_i-\lambda_i)) = \sum_i\sum_j C(n,i)B(i,j)$$

Be definition, $BC = CB = I$, then $\sum_j\sum_i C(n,i)B(i,j) = 1$, for any n. Therefore $(\tilde{\alpha}_n + \tilde{\beta}_n) = 1$.

Aggregate output under optimal allocation can be written as a function of aggregate capital K and aggregate labor L

$$Y = \bar{A}K^{\bar{\alpha}}L^{\bar{\beta}},$$

where $\bar{A} = \prod_{i=1}^{N}(\chi_i\tilde{A}_i)^{\theta_i}$ is the aggregate TFP under optimal allocation, and $\bar{\alpha} = \sum_n(\tilde{\alpha}_n\theta_n)$, $\bar{\beta} = \sum_n(\tilde{\beta}_n\theta_n)$. It can be shown easily that $\bar{\alpha} + \bar{\beta} = 1$.

The expenditure on imported goods is

$$\bar{P}_j m_{ij} = [\prod_s(\chi_s\tilde{A}_s)^{\theta_s}]\{\frac{\theta_i}{\chi_i}K^{\sum_s\theta_s\tilde{\alpha}_s}L^{\sum_s\theta_s\tilde{\beta}_s}\}\lambda_{ij} = (\frac{\theta_i\lambda_{ij}}{\chi_i})Y$$

The total expenditure on imported goods is

$$X = [\sum_{i=1}^{N}(\frac{\theta_i\lambda_i}{\chi_i})]Y.$$

The output net of imported goods is

$$Y - X = Y[1 - \sum_{i=1}^{N} (\frac{\theta_i \lambda_i}{\chi_i})].$$

Q.E.D.

B.3 Proof of Proposition 3

Firm in sector i will solve the following profit maximization problem

$$\max_{K_i, L_i, \{d_{ij}, m_{ij}\}} P_i A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} d_{i1}^{\sigma_{i1}} \dots d_{iN}^{\sigma_{iN}} m_{i1}^{\lambda_{i1}} \dots m_{iN}^{\lambda_{iN}}$$
$$- (1+\tau_i^k) R K_i - (1+\tau_i^l) w L_i - \sum_{j=1}^{N} (1+\tau_{ij}^d) P_j d_{ij} - \sum_{j=1}^{N} (1+\tau_{ij}^m) \bar{P}_j m_{ij}$$

First order conditions

$$\alpha_i (1 - \sigma_i - \lambda_i) \frac{P_i Q_i}{K_i} = R(1 + \tau_i^k) \tag{16}$$

$$(1 - \alpha_i)(1 - \sigma_i - \lambda_i) \frac{P_i Q_i}{L_i} = w(1 + \tau_i^l) \tag{17}$$

$$\sigma_{ij} \frac{P_i Q_i}{d_{ij}} = P_j (1 + \tau_{ij}^d), j = 1, \cdots, N \tag{18}$$

$$\lambda_{ij} \frac{P_i Q_i}{m_{ij}} = \bar{P}_j (1 + \tau_{ij}^m), j = 1, \cdots, N \tag{19}$$

Market clear condition for sector j

$$Y_j + \sum_{i=1}^{N} d_{ij} = Q_j$$

or

$$P_j Y_j + \sum_{i=1}^{N} P_j d_{ij} = P_j Q_j$$

Plugging in equation (18)

$$P_j Y_j + \sum_{i=1}^{N} \sigma_{ij} \frac{P_i Q_i}{(1 + \tau_{ij}^d)} = P_j Q_j$$

From final goods problem

$$P_j = \beta_j \frac{Y}{Y_j}$$

substituting for P_j and cancel for Y

$$\beta_j + \sum_{i=1}^{N} \sigma_{ij} \frac{Q_i}{(1+\tau^d_{ij})} \frac{\beta_i}{Y_i} = \frac{\beta_j}{Y_j} Q_j$$

Following the way in Jones(2011), we define $\gamma_j = \frac{\beta_j Q_j}{Y_j}$, then the following equation will solve $\{\nu_j\}$

$$\beta_j + \sum_{i=1}^{N} \frac{\sigma_{ij}}{(1+\tau^d_{ij})} \gamma_i = \gamma_j, j = 1, \cdots, N$$

Denote $\beta = [\beta_1, \cdots, \beta_N]_{N\times 1}$, $\gamma = [\gamma_1, \cdots, \gamma_N]_{N\times 1}$, $\bar{B}(i,j) = \frac{\sigma_{ij}}{1+\tau^d_{ij}}$, then

$$\gamma = (I - \bar{B})^{-1} \beta$$

Note: γ is the Domar weights–total spending on good to Y $\gamma_j = \frac{\beta_j Q_j}{Y_j} = \frac{P_j Q_j}{Y}$.

Given γ, by first order condition in equation (18), we can solve d_{ij} as

$$d_{ij} = \sigma_{ij} \frac{P_i Q_i}{P_j (1+\tau^d_{ij})} = \sigma_{ij} \frac{\gamma_i}{\gamma_j (1+\tau^d_{ij})} Q_j$$

From first order condition in equation (19),

$$\frac{m_{ij}}{d_{ij}} = \frac{1+\tau^d_{ij}}{1+\tau^m_{ij}} \frac{\lambda_{ij}}{\sigma_{ij}} \frac{P_j}{\bar{P}_j}$$

then

$$m_{ij} = \frac{\lambda_{ij}}{1+\tau^m_{ij}} \frac{P_j}{\bar{P}_j} \frac{\gamma_i}{\gamma_j} Q_j = \frac{\lambda_{ij}}{1+\tau^m_{ij}} \frac{\gamma_i}{\bar{P}_j} Y$$

From first order condition in equation (16),

$$K_i = \alpha_i (1-\sigma_i-\lambda_i) \frac{P_i Q_i}{R(1+\tau^k_i)} = \alpha_i (1-\sigma_i-\lambda_i) \frac{\gamma_j Y}{R(1+\tau^k_i)}$$

Denote

$$\delta^k_i = \alpha_i (1-\sigma_i-\lambda_i) \frac{\gamma_i}{(1+\tau^k_i)}$$

then $K_i = \delta^k_i \frac{Y}{R}$, define $\tilde{\delta}^k_i = \frac{\delta^k_i}{\sum_{i=1}^N \delta^k_i}$

$$K_i = \tilde{\delta}^k_i K.$$

Similarly, define

$$\delta_i^l = (1-\alpha_i)(1-\sigma_i-\lambda_i)\frac{\gamma_i}{(1+\tau_i^l)}$$

and $\tilde{\delta}_i^l = \frac{\delta_i^l}{\sum_{i=1}^N \delta_i^l}$ then

$$L_i = \tilde{\delta}_i^l L$$

define $\delta_{ij}^d = \frac{\sigma_{ij}}{1+\tau_{ij}^d}$ and $\delta_{ij}^m = \frac{\lambda_{ij}}{1+\tau_{ij}^m}$ then

$$d_{ij} = \delta_{ij}^d \frac{\gamma_i}{\gamma_j} Q_j$$

$$m_{ij} = \delta_{ij}^m \frac{\gamma_i}{\bar{P}_j} Y$$

Given $\{K_i, L_i, d_{ij}, m_{ij}\}$ the gross output Q_i can be computed as

$$Q_i = P_i A_i [(\tilde{\delta}_i^k K)^{\alpha_i} (\tilde{\delta}_i^l L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} \prod_{j=1}^N (\delta_{ij}^d \frac{\gamma_i}{\gamma_j} Q_j)^{\sigma_{ij}} \prod_{j=1}^N (\delta_{ij}^m \frac{\gamma_i}{\bar{P}_j} Y)^{\lambda_{ij}}$$

take log on both side

$$\log(Q_i) = \log\{P_i A_i [(\tilde{\delta}_i^k K)^{\alpha_i} (\tilde{\delta}_i^l L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i}\} + \sum_{j=1}^N \sigma_{ij} \log(\delta_{ij}^d \frac{\gamma_i}{\gamma_j}) + \sum_{j=1}^N \lambda_{ij} \log(\delta_{ij}^m \frac{\gamma_i}{\bar{P}_j})$$
$$+ \sum_{j=1}^N \sigma_{ij} \log(Q_j) + \sum_{j=1}^N \lambda_{ij} \log(Y)$$

and write in matrix form

$$q = D + \sigma q + \lambda \log(Y)$$

where q is N by 1 matrix, $q(i) = \log(Q_i)$, D is N by 1 matrix,

$$D(i) = \log\{P_i A_i [(\tilde{\delta}_i^k K)^{\alpha_i} (\tilde{\delta}_i^l L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i}\} + \sum_{j=1}^N \sigma_{ij} \log(\delta_{ij}^d \frac{\gamma_i}{\gamma_j}) + \sum_{j=1}^N \lambda_{ij} \log(\delta_{ij}^m \frac{\gamma_i}{\bar{P}_j})$$

and σ is N by N matrix, $\sigma(i,j) = \sigma_{ij}$, λ is N by 1 matrix, $\lambda(i) = \lambda_i$. then

$$q = (I-\sigma)^{-1}[D + \lambda \log(Y)]$$

Since $Y = \prod_i Y_i^{\theta_i}$,and $\gamma_j = \frac{\beta_j Q_j}{Y_j} = \frac{P_j Q_j}{Y}$. then $Y = \prod_i (\frac{\beta_i Q_i}{\gamma_i})^{\theta_i}$, taking log we have

$$\log(Y) = \sum_i \theta_i \log(\frac{\beta_i}{\gamma_i}) + \sum_i \theta_i \log(Q_i)$$

then then

$$\log(Y) = \sum_i \theta_i \log(\frac{\beta_i}{\gamma_i}) + \theta'(I-\sigma)^{-1}[D + \lambda \log(Y)]$$

where θ is N by 1 matrix, $\theta(i) = \theta_i$, then

$$\log(Y) = \frac{1}{1-\theta'(I-\sigma)^{-1}\lambda}[\sum_i \theta_i \log(\frac{\beta_i}{\gamma_i}) + \theta'(I-\sigma)^{-1}D]$$

Q.E.D.

B.4 Proof of Proposition 4 and Proposition 5

Following proposition 2 and the proof in section B.2, we know that

$$A^* = \frac{Y^*}{K^{\alpha^*}L^{1-\alpha^*}}.$$

Since in the data,

$$A = \frac{Y}{K^{\alpha}L^{1-\alpha}},$$

we can write the optimal TFP A^* as

$$A^* = A\frac{Y^*}{Y}K^{\alpha-\alpha^*}L^{\alpha^*-\alpha} = A\frac{1}{\mathbf{E}}(\frac{K}{L})^{\alpha-\alpha^*}.$$

B.5 Proof of proposition 8

To prove this proposition, we need to show that $E_t^m = E_t^d = E_t^y = 1$ in equation 4. It is equivalent to show that $\chi_{i,t} = \chi_{i,t}^*$, that is, $\chi_{i,t}$ in the data are the optimal ones.

To simplify notations, we drop the time subscript. We intend to show that χ_i satisfy the system of equations 2, which we reproduced here: $\frac{1}{\chi_i} = 1 + \frac{1}{\theta_i}\sum_s(\frac{\theta_s}{\chi_s}\sigma_{si})$. Define $\eta_i = \frac{1}{\chi_i}$, the above system of equations can be written as,

$$\eta_i = 1 + \sum_s(\eta_s \frac{\theta_s}{\theta_i}\sigma_{si}), \forall i \in \{1, ..., N\},$$

equivalently,

$$\begin{pmatrix} \eta_1 - 1 \\ \vdots \\ \eta_N - 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_{11} & \cdots & \frac{\theta_N}{\theta_1}\sigma_{N1} \\ \vdots & \ddots & \vdots \\ \frac{\theta_1}{\theta_N}\sigma_{1N} & \cdots & \sigma_{NN} \end{pmatrix}}_{\Pi} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \tag{20}$$

in which $\Pi(i,j) = \frac{\theta_j}{\theta_i}\sigma_{ji}$.

Under the assumption that the distortions are in the form of sectoral tax and subsidy and following section 3.2,

$$\begin{aligned} \eta_i &= \frac{1}{\chi_i} = \frac{Q_i}{Y_i}, \\ \sigma_{ij} &= \frac{p_j d_{ij}}{p_i Q_i}, \\ \theta_i &= \frac{p_i Y_i}{\sum_s p_s Y_s} \end{aligned}$$

Take the data η_i, σ_{ij} and θ_i back to the equation 20, the RHS can be written as

$$\begin{pmatrix} \frac{d_{11}}{Q_1} & \cdots & \frac{Y_N}{Q_N}\frac{d_{N1}}{Y_1} \\ \vdots & \ddots & \vdots \\ \frac{Y_1}{Q_1}\frac{d_{1N}}{Y_N} & \cdots & \frac{d_{NN}}{Q_N} \end{pmatrix} \begin{pmatrix} \frac{Q_1}{Y_1} \\ \vdots \\ \frac{Q_N}{Y_N} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{s=1}^N d_{s1}}{Y_1} \\ \vdots \\ \frac{\sum_{s=1}^N d_{sN}}{Y_N} \end{pmatrix} = \begin{pmatrix} \frac{Q_1 - \chi_1 Q_1}{\chi_1 Q_1} \\ \vdots \\ \frac{Q_N - \chi_N Q_N}{\chi_N Q_N} \end{pmatrix} = \begin{pmatrix} \eta_1 - 1 \\ \vdots \\ \eta_N - 1 \end{pmatrix},$$

which is the same as the LHS. *Q.E.D.*

B.6 Proof of Proposition 6

In this section, we show that $\alpha = \alpha^*$ when the economy has no distortions. From the FOCs of the decentralized problem, we know that $RK_i = p_i Q_i \alpha_i (1-\sigma_i)$ if there are no distortions in the economy. Therefore the measured capital income share in the data α_t can be written as

$$\alpha = \frac{\sum_i RK_i}{Y} = \frac{\sum_i p_i Q_i \alpha_i (1-\sigma_i)}{Y}.$$

Following the notations of the decentralized problem (see Appendix **??**), denote $\gamma_i = \frac{p_i Q_i}{Y}$ as the Domar weight for sector i. We can rewrite the above equation as,

$$\alpha = \sum_i \gamma_i \alpha_i (1-\sigma_i).$$

On the other hand, α^* can be written as

$$\begin{aligned} \alpha^* &= \sum_n \theta_n \sum_i (\alpha_i (1-\sigma_i) C_{ni}) \\ &= \sum_i \alpha_i (1-\sigma_i) \sum_n \theta_n C_{ni}, \end{aligned}$$

where $C = (I - \Omega)^{-1}$. We've shown in the decentralization problem that $\gamma_i = \sum_n \theta_n C_{ni}$ which implies $\alpha = \alpha^*$.

B.7 Proof of Proposition 9

We can write the allocative efficiency with input-output linkages as

$$\begin{aligned} \log \mathbf{E}_t^{wio} = & \sum_i \alpha_i (1 - \sigma_i - \lambda_i)(\sum_n \theta_n C_{n,i}) \log(\frac{\chi_{K,i}}{\chi^*_{K,i}}) \\ & + \sum_i (1 - \alpha_i)(1 - \sigma_i - \lambda_i)(\sum_n \theta_n C_{n,i}) \log(\frac{\chi_{L,i}}{\chi^*_{L,i}}) \\ = & \sum_n \theta_n \sum_i C_{n,i}(1 - \lambda_i - \sigma_i)\alpha_i \log(\frac{\chi_{K,i}}{\chi^*_{K,i}}) \\ & + \sum_n \theta_n \sum_i C_{n,i}(1 - \lambda_i - \sigma_i)(1 - \alpha_i) \log(\frac{\chi_{L,i}}{\chi^*_{L,i}}) \end{aligned}$$

Notice that we know

$$\begin{aligned} \sum_i C_{n,i}(1 - \lambda_i - \sigma_i) &= 1 \ \forall n \\ \sum_n \theta_n &= 1. \end{aligned}$$

Because $C_{n,i}(1 - \lambda_i - \sigma_i) > 0$ for all i, it must be $\sum_i C_{n,i}(1 - \lambda_i - \sigma_i) \in (0, 1)$. Then

$$\sum_n \theta_n \sum_i C_{n,i}(1 - \lambda_i - \sigma_i)\alpha_i \log(\frac{\chi_{K,i}}{\chi^*_{K,i}}) < \sum_i \alpha_i \log(\frac{\chi_{K,i}}{\chi^*_{K,i}}),$$
$$\sum_n \theta_n \sum_i C_{n,i}(1 - \lambda_i - \sigma_i)(1 - \alpha_i) \log(\frac{\chi_{L,i}}{\chi^*_{L,i}}) < \sum_i (1 - \alpha_i) \log(\frac{\chi_{L,i}}{\chi^*_{L,i}}).$$

Therefore,

$$\log \mathbf{E}_t^{wio} < \log \mathbf{E}_t^{woio}.$$

B.8 Proof of Proposition 10

The FOCs of the planner's problem give

$$\begin{aligned}\omega_i(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} &= \frac{K_i/\alpha_i}{\sum_i(K_i/\alpha_i)},\\ \omega_i(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} &= \frac{L_i/(1-\alpha_i)}{\sum_i[L_i/(1-\alpha_i)]}.\end{aligned}$$

To simplify notations, we denote $\tilde{K}_i = K_i/\alpha_i$, $\tilde{L}_i = L_i/(1-\alpha_i)$ and define $\tilde{K} = \sum_i \tilde{K}_i$ and $\tilde{L} = \sum_i \tilde{L}_i$. It is clear that, from the FOCs, $\frac{\tilde{K}_i}{\tilde{L}_i} = \frac{\sum_i \tilde{K}_i}{\sum_i \tilde{L}_i} = \frac{\tilde{K}}{\tilde{L}}$. We can rewrite K_i and L_i using the production functions as

$$\begin{aligned}K_i &= (\alpha_i\tilde{K})\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1},\\ L_i &= (1-\alpha_i)\tilde{L}\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}.\end{aligned}$$

Given $\sum_i K_i = K, \sum_i L_i = L$, we can solve $Y, \tilde{K}, \tilde{L}$ with the system of three equations

$$\begin{aligned}K &= \sum_i\{(\alpha_i\tilde{K})\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\},\\ L &= \sum_i\{(1-\alpha_i)\tilde{L}\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\},\\ Y^{\rho-1} &= \sum_i \omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}.\end{aligned}$$

In particular,

$$\begin{aligned}\frac{K}{\tilde{K}} &= \sum_i\{\alpha_i\frac{\omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\alpha_j\tilde{K})^{\alpha_j}[(1-\alpha_j)\tilde{L}]^{1-\alpha_j}\}^{\rho-1}}\},\\ \frac{L}{\tilde{L}} &= \sum_i\{(1-\alpha_i)\frac{\omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\alpha_j\tilde{K})^{\alpha_j}[(1-\alpha_j)\tilde{L}]^{1-\alpha_j}\}^{\rho-1}}\},\end{aligned}$$

and $\frac{K}{\tilde{K}} + \frac{L}{\tilde{L}} = 1$.

Denote $\bar{\alpha} = \frac{K}{\tilde{K}}$, then $\frac{L}{\tilde{L}} = 1-\bar{\alpha}$, and $\bar{\alpha}$ solves the following equation

$$\bar{\alpha} = \sum_i\{\alpha_i\frac{\omega_i^{\rho}\{A_i(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\}, \tag{21}$$

and the output under optimal allocation is

$$Y^* = \{\sum_j \omega_j^\rho \{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{\rho-1}}.$$

If we replace, in the previous equation, A_i with $\frac{Y_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}$ and Y_i with $(\frac{P_iY_i/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y$, we can rewrite the allocative efficiency as

$$\begin{aligned}
\mathbf{E} = \frac{Y}{Y^*} &= \{\sum_j \omega_j^\rho \{\frac{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}}{K_j^{\alpha_j}L_j^{1-\alpha_j}}(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}, \\
&= \{\sum_j \{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}},
\end{aligned}$$

which means that E is a function of $\bar{\alpha}$, the expenditure share $\frac{P_jY_j}{PY}$ in the data, capital allocation $\frac{K_j}{K}$ and labor allocation $\frac{L_j}{L}$ in the data. All the other measures, except for $\bar{\alpha}$ are clearly unit-less. We show next that so is $\bar{\alpha}$. By replacing A_i in equation 21, we can write the equation as the following,

$$\begin{aligned}
\bar{\alpha} &= \sum_i \{\alpha_i \frac{\omega_i^\rho \{A_i(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho \{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\omega_i^\rho \{\frac{Y_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho \{\frac{Y_j}{K_j^{\alpha_j}L_j^{1-\alpha_j}}(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\omega_i^\rho \{(\frac{P_iY_i/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y(\frac{\alpha_i}{\bar{\alpha}}K/K_i)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L/L_i]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho \{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y(\frac{\alpha_j}{\bar{\alpha}}K/K_j)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L/L_j]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{Y^{\rho-1}\{(\frac{P_iY_i}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_i}{\bar{\alpha}}K/K_i)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L/L_i]^{1-\alpha_i}\}^{\rho-1}}{\sum_j Y^{\rho-1}\{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j}{\bar{\alpha}}K/K_j)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L/L_j]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\{(\frac{P_iY_i}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_i}{\bar{\alpha}}\frac{K}{K_i})^{\alpha_i}(\frac{1-\alpha_i}{1-\bar{\alpha}}\frac{L}{L_i})^{1-\alpha_i}\}^{\rho-1}}{\sum_j\{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j}{\bar{\alpha}}\frac{K}{K_j})^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}\frac{L}{L_j}]^{1-\alpha_j}\}^{\rho-1}}\},
\end{aligned}$$

which is clear that $\bar{\alpha}$ only depends on the expenditure share $\frac{P_jY_j}{PY}$ in the data, capital allocation $\frac{K_j}{K}$ and labor allocation $\frac{L_j}{L}$ in the data. Note that $\bar{\alpha}$ is unitless.

In addition, one can easily verify that $\bar{\alpha} = \sum_i \frac{P_iY_i}{PY}\alpha_i$ and the following allocation of capital and labor

$$\frac{K_i}{K} = \frac{\frac{P_iY_i}{PY}\alpha_i}{\sum_i \frac{P_iY_i}{PY}\alpha_i} = \frac{\frac{P_iY_i}{PY}\alpha_i}{\bar{\alpha}} \quad \text{and} \quad \frac{L_i}{L} = \frac{\frac{P_iY_i}{PY}(1-\alpha_i)}{\sum_i \frac{P_iY_i}{PY}(1-\alpha_i)} = \frac{\frac{P_iY_i}{PY}(1-\alpha_i)}{1-\bar{\alpha}}$$

solve equations 9 and 10 and therefore are the optimal allocation. We denote $\alpha^* = \sum_i \frac{P_iY_i}{PY}\alpha_i$, $\chi_i^{k*} = \frac{\frac{P_iY_i}{PY}\alpha_i}{\sum_i \frac{P_iY_i}{PY}\alpha_i}$, and $\chi^{l*} = \frac{\frac{P_iY_i}{PY}(1-\alpha_i)}{\sum_i \frac{P_iY_i}{PY}(1-\alpha_i)}$. Note that the optimal allocation of capital and labor does not depend on the elasticity of substitution ρ. We can rewrite the allocative efficiency as

$$\begin{aligned}
\mathbf{E} &= \{\sum_j \{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}} \\
&= \{\sum_j \{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}
\end{aligned}$$

B.9 CES production system with input-output linkages

There are N sectors in the economy, each sector produces a good Y_i using capital, labor and other sectors' output, other countries' goods such that,

$$Q_i = A_i(K_i^{\alpha_i}L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} d_{i1}^{\sigma_{i1}} \dots d_{iN}^{\sigma_{iN}} m_{i1}^{\lambda_{i1}} \dots m_{iN}^{\lambda_{iN}},$$

where $\sum_{j=1}^N \sigma_{ij} = \sigma_i$, $\sum_{j=1}^N \lambda_{ij} = \lambda_i$. The final good is the CES aggregation of all intermediate inputs

$$Y_i = Q_i - \sum_{j=1}^N d_{ji}$$
$$Y = (\sum_i Y_i^{1-\frac{1}{\rho}})^{\frac{\rho}{\rho-1}}.$$

The planner's problem reads

$$\max_{\{K_i, L_i, d_{ij}, m_{ij}\}} C = Y - X \;\; \text{s.t.} \; \sum_i K_i = K, \; \sum_i L_i = L, \; X = \sum_i \sum_j \bar{P}_j m_{ij}.$$

FOCs of the planner's problem are

$$\frac{\partial C}{\partial K_i} = (\sum_i Y_i^{1-\frac{1}{\rho}})^{\frac{1}{\rho-1}} Y_i^{-\frac{1}{\rho}} \frac{Q_i}{K_i} \alpha_i (1-\sigma_i-\lambda_i) = \lambda \tag{22}$$

$$\frac{\partial C}{\partial L_i} = (\sum_i Y_i^{1-\frac{1}{\rho}})^{\frac{1}{\rho-1}} Y_i^{-\frac{1}{\rho}} \frac{Q_i}{L_i} (1-\alpha_i)(1-\sigma_i-\lambda_i) = \eta \tag{23}$$

$$\frac{\partial C}{\partial d_{ij}} = (\sum_i Y_i^{1-\frac{1}{\rho}})^{\frac{1}{\rho-1}} (Y_i^{-\frac{1}{\rho}} \frac{\partial Y_i}{\partial d_{ij}} + Y_j^{-\frac{1}{\rho}} \frac{\partial Y_j}{\partial d_{ij}}) = 0, \tag{24}$$

$$\frac{\partial C}{\partial m_{ij}} = (\sum_i Y_i^{1-\frac{1}{\rho}})^{\frac{1}{\rho-1}} Y_i^{-\frac{1}{\rho}} \frac{Q_i}{m_{ij}} \lambda_{ij} - \bar{P}_j = 0, \tag{25}$$

where

$$\frac{\partial Y_i}{\partial d_{ij}} = \frac{Q_i}{d_{ij}} \sigma_{ij} - I_{\{i=j\}}; \ \frac{\partial Y_j}{\partial d_{ij}} = \frac{Q_j}{d_{jj}} \sigma_{jj} I_{\{i=j\}} - 1$$

FOC 24 implies that

$$\begin{aligned} d_{ij} &= (\frac{Y_i}{Y_j})^{-\frac{1}{\rho}} \sigma_{ij} Q_i, \\ Y_j &= Q_j - \sum_{i=1}^{N} d_{ij} = Q_j - \sum_{i=1}^{N} (\frac{Y_i}{Y_j})^{-\frac{1}{\rho}} \sigma_{ij} Q_i, \\ Q_j &= Y_j [1 + \sum_i (\frac{Y_i}{Y_j})^{1-\frac{1}{\rho}} \sigma_{ij} \frac{Q_i}{Y_i}]. \end{aligned}$$

Denote $Y_j := \chi_j Q_j$. Given $\{Y_i\}$, $\{\chi_i\}$ solve the following system of equations

$$\frac{1}{\chi_i} = 1 + \sum_s ((\frac{Y_s}{Y_i})^{1-\frac{1}{\rho}} \frac{1}{\chi_s} \sigma_{si}).$$

And we have

$$d_{ij} = (\frac{Y_i}{Y_j})^{1-\frac{1}{\rho}} \frac{\chi_j}{\chi_i} \sigma_{ij} Q_j.$$

Denote $d_{ij} := \gamma_{ij} Q_j$, we have $\gamma_{ij} = (\frac{Y_i}{Y_j})^{1-\frac{1}{\rho}} \frac{\chi_j}{\chi_i} \sigma_{ij}$, and the relationship between γ_{ij} and χ_i is

$$\chi_i = 1 - \sum_s \gamma_{si}.$$

FOC 25 implies

$$
\begin{aligned}
m_{ij} &= (\frac{Y}{Y_i})^{\frac{1}{\rho}} Q_i \frac{\lambda_{ij}}{\bar{P}_j}, \\
m_{ij} &= [\sum_s (\frac{\chi_s}{\chi_i} Q_s)^{1-\frac{1}{\rho}}]^{\frac{1}{1-\rho}} Q_i^{1+\frac{1}{\rho}} \frac{\lambda_{ij}}{\bar{P}_j}.
\end{aligned}
$$

FOCs 22 and 23 solve optimal capital and labor allocation

$$
\begin{aligned}
(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} &= \frac{\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)}}{\sum_i (\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)})}, \\
(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} &= \frac{\chi_i \frac{L_i}{(1-\alpha_i)(1-\sigma_i-\lambda_i)}}{\sum_i (\chi_i \frac{L_i}{(1-\alpha_i)(1-\sigma_i-\lambda_i)})}.
\end{aligned}
$$

Denote $\tilde{K}_i = \chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)}$ and $\tilde{L}_i = \chi_i \frac{L_i}{(1-\alpha_i)(1-\sigma_i-\lambda_i)}$, we have $\frac{\tilde{K}_i}{\tilde{L}_i} = \frac{\sum_i \tilde{K}_i}{\sum_i \tilde{L}_i}$. Define $\tilde{K} = \sum_i \tilde{K}_i$ and $\tilde{L} = \sum_i \tilde{L}_i$.

We can rewrite Q_i as

$$
Q_i = \tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} K_i^{\alpha_i} L_i^{1-\alpha_i},
$$

where $\tilde{A}_i = A_i \prod_j [(\frac{Y_j}{Y_i})^{\frac{1}{\rho}} \sigma_{ij}]^{\sigma_{ij}} \prod_j [\frac{\lambda_{ij}}{\bar{P}_j}]^{\lambda_{ij}} (\frac{Y}{Y_i})^{\frac{\lambda_i}{\rho}}$. Therefore

$$
\begin{aligned}
Y_i &= \tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\chi_i K_i)^{\alpha_i} (\chi_i L_i)^{1-\alpha_i} \\
&= \tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} [\alpha_i(1-\sigma_i-\lambda_i)\tilde{K}_i]^{\alpha_i} [(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}_i]^{1-\alpha_i} \\
&= \tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} [\frac{\alpha_i(1-\sigma_i-\lambda_i)\tilde{K}}{(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}}]^{\alpha_i-1} [\alpha_i(1-\sigma_i-\lambda_i)\tilde{K}_i].
\end{aligned}
$$

We can rewrite $\chi_i K_i, \chi_i L_i$ as

$$
\begin{aligned}
\chi_i K_i &= [\alpha_i(1-\sigma_i-\lambda_i)\tilde{K}][\frac{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i} [(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}, \\
\chi_i L_i &= [(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}][\frac{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i} [(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}.
\end{aligned}
$$

Because $\sum_i K_i = K, \sum_i L_i = L$, we can solve $Y, \tilde{K}, \tilde{L}$ using the following three equations

$$\begin{aligned}
K &= \sum_i \{ \frac{[\alpha_i(1-\sigma_i-\lambda_i)\tilde{K}]}{\chi_i} [\frac{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\}, \\
L &= \sum_i \{ \frac{[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]}{\chi_i} [\frac{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\}, \\
Y^{\rho-1} &= \sum_i \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
\frac{K}{\tilde{K}} &= \sum_i \{ \frac{[\alpha_i(1-\sigma_i-\lambda_i)]}{\chi_i} \frac{\{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}\}, \\
\frac{L}{\tilde{L}} &= \sum_i \{ \frac{[(1-\alpha_i)(1-\sigma_i-\lambda_i)]}{\chi_i} \frac{\{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} (\alpha_i(1-\sigma_i-\lambda_i)\tilde{K})^{\alpha_i}[(1-\alpha_i)(1-\sigma_i-\lambda_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}\}.
\end{aligned}$$

Notice that if $(1-\sigma_i-\lambda_i) \neq \chi_i$, then $\frac{K}{\tilde{K}} + \frac{L}{\tilde{L}} \neq 1$, which is different from the result without input-output linkages.

In the above equations, replace $\alpha_K^* = \frac{K}{\tilde{K}}, \alpha_L^* = \frac{L}{\tilde{L}}$. Thus α_K^* and α_L^* solve the follow equations,

$$\begin{aligned}
\alpha_K^* &= \sum_i \{ \frac{[\alpha_i(1-\sigma_i-\lambda_i)]}{\chi_i} \frac{\{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} ((1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K)^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_i \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} ((1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K)^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}}\} \\
\alpha_L^* &= \sum_i \{ \frac{[(1-\alpha_i)(1-\sigma_i-\lambda_i)]}{\chi_i} \frac{\{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} ((1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K)^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_i \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} ((1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K)^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}}\}.
\end{aligned}$$

The output under optimal allocation is

$$
\begin{aligned}
Y^* &= \{\sum_i \{\tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}}[(1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K]^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}\}^{\frac{1}{\rho-1}}, \\
&= \{\sum_i \{\frac{Q_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}[(1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K]^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}\}^{\frac{1}{\rho-1}} \\
&= \{\sum_i \{\frac{Y_i/\chi_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}[(1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}K]^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}L]^{1-\alpha_i}\}^{\rho-1}\}^{\frac{1}{\rho-1}} \\
&= \{\sum_i \{[\frac{\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)}}{\sum_i(\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)})}]^{\frac{\rho}{\rho-1}}\frac{Y}{\chi_i}[(1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}\frac{K}{K_i}]^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}\frac{L}{L_i}]^{1-\alpha_i}\}^{\rho-1}\}^{\frac{1}{\rho-1}}.
\end{aligned}
$$

The second step of the above equations uses equation $Q_i = \tilde{A}_i^{\frac{1}{1-\sigma_i-\lambda_i}} K_i^{\alpha_i} L_i^{1-\alpha_i}$ and the last step uses equation $(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} = \frac{\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)}}{\sum_i(\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)})}$.

We derive the allocative efficiency as

$$
\frac{Y^*}{Y} = \{\sum_i \{[\frac{\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)}}{\sum_i(\chi_i \frac{K_i}{\alpha_i(1-\sigma_i-\lambda_i)})}]^{\frac{\rho}{\rho-1}}\frac{1}{\chi_i}[(1-\sigma_i-\lambda_i)\frac{\alpha_i}{\alpha_K^*}\frac{K}{K_i}]^{\alpha_i}[(1-\sigma_i-\lambda_i)\frac{(1-\alpha_i)}{\alpha_L^*}\frac{L}{L_i}]^{1-\alpha_i}\}^{\rho-1}\}^{\frac{1}{\rho-1}}.
$$